Geometry of Schemes Problems

Week 2

I-30

A scheme is irreducible iff all (nonempty) open sets are dense.

Proof. Every open subset is dense $\iff$ every nonempty open subset meets every other nonempty open set $\iff$ some element is missing from the union of any pair of proper closed subsets.

$\square$

I-31

Let $X = Spec(R)$. X is reduced and irreducible iff R is a domain. X is irreducible iff R has a unique minimal prime, i.e. if the nilradical is prime.

Proof. **Unique minimal prime $\iff$ nilradical is prime**

Suppose R has a unique minimal prime. Then

$$\sqrt{(0)} = \bigcap_{\text{primes P}} P = \bigcap_{\text{minimal primes P}} P$$

Thus the nilradical is prime.

On the other hand, the nilradical belongs to every prime ideal. Thus if it is prime then it is a prime ideal belonging to every other prime ideal, hence it is minimal and no other prime is minimal.

R irreducible $\iff$ nilradical is prime

To see that this is equivalent to X being irreducible, suppose the nilradical is prime. Let $g, f \in R \setminus \sqrt{(0)}$. Then $X_f \cap X_g \neq \emptyset$ since $\sqrt{(0)} \in X_f \cap X_g$. If $g \in \sqrt{(0)}$, then $X_g = \emptyset$. So any nonempty basis element is dense, and therefore the same is true for any open set (any open set is a union of dense basis elements).

Suppose now that the nilradical is not prime. Let $f, g \in R \setminus \sqrt{(0)}$ such that $fg \in \sqrt{(0)}$. Then $X_f \cap X_g = \emptyset$ since each prime contains $\sqrt{(0)}$, hence contains fg and therefore at least one of f and g. However neither X_f nor X_g is empty, for if so f or g would have to be in the nilradical.

X reduced and irreducible $\iff$ R is a domain

Since X is irreducible, the nilradical of R is prime, and since X is reduced R has trivial nilradical. Thus (0) is prime and R is a domain. On the other hand for a domain R, (0) is prime. Hence R is nilpotence-free so X is reduced. In addition, $(0) \in X_f \forall f \neq 0$ so that (0) is in all nonempty basic open sets, and therefore in all nonempty open sets. Therefore any two nonempty open sets meet one another, and so X is irreducible. $\square$

I-34

An arbitrary scheme X is irreducible iff every open affine subset is irreducible. If $|X|$ is connected as a topological space, this holds iff every local ring of $\mathcal{O}_X$ has a unique minimal prime.

Proof. Suppose every affine open subset of X is irreducible, and let U and V be nonempty open sets. It must be shown that they meet one another. Let C_1 and C_2 be two affine open charts so that U meets C_1 and V meets C_2 (both necessarily in open sets).

Case 1: non-disjoint charts $C_1 \cap C_2 \neq \emptyset$, their intersection is an open set dense in both C_1 and C_2. Hence U and V both meet the intersection, and thus $U \cap C_2 \neq \emptyset$. Then $U \cap V \cap C_2 \neq \emptyset$ because C_2 is irreducible and meets U and V.

Case 2: disjoint charts

If $C_1 \cap C_2 = \emptyset$, then by I-33, $C_1 \sqcup C_2$ is again an affine open. $U \cap V \cap (C_1 \sqcup C_2)$ is therefore nonempty.

Suppose then that X is irreducible, and let C be an open affine chart with U and V nonempty open subsets of C. Since C is open, U and V are nonempty open subsets of X, so $U \cap V \neq \emptyset$ by irreducibility of X.

If X is irreducible then if $p \in X$ belongs to an affine open $U = Spec(R)$, U is irreducible. By I-31, R has a unique minimal prime q. Then $\mathcal{O}_{X,p} \cong R_p$ and the primes of this localization are the primes of R contained in p. Of these, q is the unique minimal prime in R, so its image is the unique minimal prime in $\mathcal{O}_{X,p}$.

According to Stackexchange, the converse is false.

$\square$