Geometry of Schemes Problems

Week 4

II-6

Find the images under the map $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$ of the following points.

a

$(x - \sqrt{2}, y - \sqrt{2})$

Here the Galois orbit of the point $(\sqrt{2}, \sqrt{2})$ contains itself and $(-\sqrt{2}, -\sqrt{2})$. Therefore the pre-image is a prime ideal containing both $(x^2 - 2)$ and $(y^2 - 2)$, but as in the discussion of complex numbers over $\mathbb{R}$, $(x^2 - 2, y^2 - 2)$ is not prime. However, since these points clearly satisfy x=y, we may consider the ideal $(x^2 - 2, x - y) \subset \mathbb{Q}[x, y]$. This ideal is certainly prime since the quotient is $\mathbb{Q}(\sqrt{2})$, and the image of x and y are equal and can be chosen to be either $\pm\sqrt{2}$ as desired. Hence this ideal is the pre-image of $(x-\sqrt{2}, y-\sqrt{2})$ in $\overline{\mathbb{Q}}[x, y]$.

b

$(x - \sqrt{2}, y - \sqrt{3})$

The Galois orbit of this point is $(x \pm \sqrt{2}, y \pm \sqrt{3})$ where the left and right signs may be chosen independently. The ideal $(x^2 - 2, y^2 - 3)$ is prime and therefore the pre-image, since it gives rise to the biquadratic extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.

c

$(x - \zeta, y - \zeta^{-1})$ where ζ is a p^{th} root of unity for p prime.

If $\zeta = 1$ this is trivial, and otherwise ζ is primitive. The Galois group for a primitive root of unity is $\mathbb{Z}/p - 1$ (the multiplicative group of $\mathbb{Z}/p$). On a single primitive root (out to a group automorphism of $\mathbb{Z}/p - 1$) we may assume that a generator is given by $\zeta \mapsto \zeta^2$. This action carries over to ζ^{-1}, so the Galois orbit of the point consists of points (ζ^n, ζ^{-n}) where $n = 1, ...p - 1$. As in part (**a**), denoting the $p - 1^{st}$ cyclotomic polynomial $f_{p-1}(x)$, we have a pre-image including points $(f_{p-1}(x), f_{p-1}(y))$, but this ideal is again not prime. Accounting for the requirement that the images of x and y be inverse, we have the ideal $(f_{p-1}(x), xy - 1)$ which is prime since its residue ring is $\mathbb{Q}(\zeta)$.

d

$(\sqrt{2}x - \sqrt{3}y)$

The orbit of $(\sqrt{2}x - \sqrt{3}y)$ contains $(\sqrt{2}x + \sqrt{3}y)$, so the image of this point will be a prime p mapping into the intersection of these two prime ideals, which contains the product ideal $(2x^2 - 3y^2)$. However since these two primes are both principle, they are generated by irreducible polynomials and therefore intersection and product coincide. Hence the image of $(\sqrt{2}x - \sqrt{3}y)$ is a prime contained in $(2x^2 - 3y^2)$. The image also contains $2x^2 - 3y^2 = (\sqrt{2}x - \sqrt{3}y)(\sqrt{2}x + \sqrt{3}y)$, completing the proof.

e

$(\sqrt{2}x - \sqrt{3}y - 1)$

Here there are four Galois conjugates, $(\pm\sqrt{2}x \pm \sqrt{3}y - 1)$. Their product is

$$(4x^4 + 9y^4 - 12x^2y^2 - 4x^2 - 6y^2 + 1)$$

So the image of $(\sqrt{2}x - \sqrt{3}y - 1)$ is the pre-image of the intersection of the 4 conjugate ideals, which contains this quartic.

II-7

Classify the subschemes of $\mathbb{A}^2{}_{\mathbb{Q}}$ as reducible, irreducible but not absolutely irreducible, or absolutely irreducible.

a

$V(x^2 - y^2)$

Over $\mathbb{Q}$, $x^2 - y^2 = (x+y)(x-y)$ so this subscheme is reducible.

b

$V(x^2 + y^2)$

This is irreducible over $\mathbb{Q}$, but factors into $(x+iy)(x-iy)$ over $\overline{Q}$. Hence the pre-image of $V(x^2+y^2)$ is not irreducible, since it is the union of $V(x+iy)$ and $V(x-iy)$. This is therefore irreducible but not absolutely irreducible.

c

$V(x^2 + y^2 - 1)$

As before this is irreducible over $\mathbb{Q}$, but it remains irreducible over $\overline{\mathbb{Q}}$. To see this suppose toward a contradiction it were not, and by scaling the leading terms of the factors let $(x + ay + c_1)(x + by + c_2) = x^2 + y^2 - 1$. We calculate

$$\begin{aligned} axy + bxy &= 0 \\ a &= -b \\ aby^2 &= -a^2y^2 = y^2 \\ a &= \pm i \\ c_1x + c_2x &= ic_1y - ic_2y = 0 \\ c_1 + c_2 &= c_1 - c_2 = 0 \end{aligned}$$

Which is contradiction since $(x+iy)(x-iy) = (x^2+y^2)$.

d

$V(x + y, xy - 2)$

This ideal is prime (indeed, maximal) since the quotient of $\mathbb{Q}$ by it is $\mathbb{Q}(\sqrt{-2})$. Over $\overline{\mathbb{Q}}$ $x = -y$ and $xy = 2$ describe the two closed points $(\pm i\sqrt{2}, \mp i\sqrt{2})$, and the subscheme is therefore reducible.

e

$V(x^2 - 2y^2, x^3 + 3y^3)$

Consider the points of $\mathbb{A}^2_{\overline{Q}}$ contained in the image. If a prime ideal contains both $x^2 - 2y^2$ and $x^3 + 3y^3$, then it contains one of the pair $y \pm \frac{1}{\sqrt{2}}x$ and one of the triple $y + \zeta^n \frac{1}{\sqrt[3]{3}}x$ where ζ is a primitive third root of unity and $n = 1, 2, 3$. Subtracting we see such an ideal contains an element ax where a is some algebraic number, and therefore must be the ideal (x, y). So the image of this ideal is the singleton set of the origin, which is irreducible. The ideal is therefore absolutely irreducible.