Geometry of Schemes Problems

Week 5

II-14

Let $C \subset \mathbb{A}^n_K$ be the curve given by the ideal $J = (x_2 - x_1^2, x_3 - x_1^3, ...)$, so that C is parameterized $f(t) = (t, t^2, ...t^n)$ for $t \in K$. Let X_t be the three-point subscheme $\{f(0), f(t), f(2t)\}$, and consider the limit as $t \to 0$.

a

Show that the limit scheme is $X_0 = spec\big(K[x_1, ...x_n]/(x_2 - x_1^2, x_1x_2, x_3, ...x_n)\big)$, isomorphic to the triple point $K[x]/(x^3)$.

Proof. Denote I_t the ideal of functions vanishing on X_t. Then we have $x_1(x_1 - t)(x_1 - 2t) \in I_t$, so passing to the limit we conclude $x_1^3 = x_1x_2 = x_3 \in I_0$. This shows that the ideal for X_0 must contain the desired elements. But since $K[x_1, ...x_n]/(x_2 - x_1^2, x_1x_2, x_3, ...x_n)$ is a 3-dimensional vector space over K, this is then the desired scheme.

The map $\phi : K[x_1, x_2, ...x_n] \to K[x]/(x^3)$ given by $\phi(x_1) = x$, $\phi(x_2) = x^2$, and $\phi(x_i) = 0 \forall i \geq 3$ factors as an isomorphism through the $K[x_1, ...x_n]/(x_2 - x_1^2, x_1x_2, x_3, ...x_n)$, and this therefore induces an isomorphism on schemes $\phi^* : spec(K[x]/(x^3)) \to spec\big(K[x_1, ...x_n]/(x_2 - x_1^2, x_1x_2, x_3, ...x_n)\big)$

□

b

Show that X_0 is not contained in the tangent line to C at the origin, but is contained in the $x_1 - x_2$ plane.

Proof. The tangent line to the curve has slope $(1, 0, 0, ...)$, hence is the zero locus of the ideal $(x_2, x_3, ...x_n)$. The triple point is not contained in the line because the reverse containment fails for the ideals. That is, x_2 is not in $(x_2 - x_1^2, x_1x_2, x_3, ...x_n)$. However, the reverse containment certainly holds for the ideal $(x_3, ...x_n)$ corresponding to the $x_1 - x_2$ plane.

□