Geometry of Schemes Problems

Week 6

II-31

Let B be the affine scheme $\mathrm{Spec}(K[s,t,u]/(su-t^2))$ and $\mathscr{X}$ $=\mathrm{Spec}(K[x,y])$ the affine plane. Let $p \in B$ be the prime (s,t,u). Let $\phi : \mathscr{X} \to B$ be the map dual to $\phi^\# : K[s,t,u]/(su-t^2) \to K[x,y]$ by $s \mapsto x^2$, $u \mapsto y^2$, and $t \mapsto xy$. Show that ϕ is not flat over p.

Proof. The pre-image of p in B is identified with $\mathrm{Spec}(K \otimes_R K[x,y]/\phi^\#(s,t,u)K[x,y])$ where $R = K[s,t,u]/(su-t^2)$, which in particular contains a copy of K. The image of (s,t,u) is an ideal. Thus we can move all scalars to the righthand side and simplify the quotient so that $\phi^{-1}(p) = \mathrm{Spec}(K[x,y]/(x^2,xy,y^2))$

This is certainly not the limit of nearby fibers, since it is the spectrum of a 3-dimensional k-algebra. The pre-image of any other closed point q, say $q = (s-a, t-\sqrt{ac}, u-c)$ where a and c are squares in K, is $\mathrm{Spec} K[x,y]/(x^2-a, xy-\sqrt{ac}, y^2-c)$. But the ideal $(x^2-a, xy-\sqrt{ac}, y^2-c) = (x-\sqrt{a}, y-\sqrt{c}) \cap (x+\sqrt{a}, y+\sqrt{c})$ since these ideals are maximal and thus comaximal. Then by Chinese Remainder Theorem, $K[x,y]/(x^2-a, xy-\sqrt{ac}, y^2-c)$ is a 2-dimensional K-vector space. The pre-image consists of two opposite points, corresponding to the fact that ϕ is equivalent to the quotient of $K[x,y]$ by the self-map $x \mapsto -x$, $y \mapsto -y$. $\square$