Geometry of Schemes: I.1 Affine Schemes

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Problem 20

Describe the points and the sheaf of functions of each of the following schemes.[a]

1. $X_1 = \operatorname{Spec} \mathbb{C}[x]/(x^2)$

   Proof. Prime ideals of $\mathbb{C}[x]/(x^2)$ are images of prime ideals in $\mathbb{C}[x]$ containing (x^2). $\mathbb{C}[x]$ is a PID and hence has unique factorization. Thus $x^2 = x \cdot x$, proving that (x) is the only prime ideal containing (x^2). Thus $\operatorname{Spec} \mathbb{C}[x]/(x^2) = \{(x)\}$, which must have the trivial topology. The sheaf of functions takes values in the original ring $\mathbb{C}[x]/(x^2)$, sending the only open set to the whole ring $\mathbb{C}[x]/(x^2)$. ∎

2. $X_2 = \operatorname{Spec} \mathbb{C}[x]/(x^2 - x)$

   Proof. As before, we can look at the factorization $x^2 - x = x(x-1)$ to see that the only prime ideals containing $x^2 - x$ are (x) and $(x-1)$, thus $\operatorname{Spec} \mathbb{C}[x]/(x^2-x) = \{(x), (x-1)\}$. Both $\{(x)\}$ and $\{(x-1)\}$ are open sets – both are in fact basic open sets since $(X_2)_x = \{(x-1)\}$ and $(X_2)_{x-1} = \{(x)\}$. Thus X_2 has the discrete topology. The sheaf of functions on X_2 takes values in $\mathbb{C}[x]/(x^2 - x)$, and can be determined by its values on the basic open sets.

$$\mathscr{O}_{X_2}\big((x)\big) = \mathscr{O}_{X_2}\big((X_2)_{x-1}\big) = \big(\mathbb{C}[x]/(x^2-x)\big)_{x-1} = (\mathbb{C}[x]/(x))_{x-1} = (\mathbb{C})_{-1} = \mathbb{C}.$$

$$\mathscr{O}_{X_2}\big((x-1)\big) = \mathscr{O}_{X_2}\big((X_2)_{x}\big) = \big(\mathbb{C}[x]/(x^2-x)\big)_{x} = (\mathbb{C}[x]/(x-1))_{x} = (\mathbb{C})_{1} = \mathbb{C}.$$

   Note that in these cases, these are actually the stalk of X_2 at (x) and $(x-1)$, respectively since our multiplicatively closed system is a single open set. ∎

3. $X_3 = \operatorname{Spec} \mathbb{C}[x]/(x^3 - x^2)$

   Proof. Since $x^3 - x^2 = x^2(x-1)$, our spectrum is the same as before: $\operatorname{Spec} \mathbb{C}[x]/(x^3-x^2) = \{(x), (x-1)\}$. As before, we have a discrete topology. The sheaf of functions on X_3 takes values in $\mathbb{C}[x]/(x^3 - x^2)$, and can be determined by its values on the basic open sets.

$$\mathscr{O}_{X_3}\big((x-1)\big) = \mathscr{O}_{X_3}\big((X_3)_{x}\big) = \big(\mathbb{C}[x]/(x^3-x^2)\big)_{x} = (\mathbb{C}[x]/(x-1))_{x} = \mathbb{C}_1 = \mathbb{C}.$$

$$\mathscr{O}_{X_3}\big((x)\big) = \mathscr{O}_{X_3}\big((X_3)_{x-1}\big) = \big(\mathbb{C}[x]/(x^3-x^2)\big)_{x-1} = \big(\mathbb{C}[x]/(x^2)\big)_{x-1}$$

   However, $\big(\mathbb{C}[x]/(x^2)\big)_{x-1} \cong \mathbb{C}[x]/(x^2)$ since $x-1$ is already a unit $- -(x-1)(x+1) = 1$. ∎

4. $X_4 = \operatorname{Spec} \mathbb{R}[x]/(x^2 + 1)$

   Proof. We know that $\mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$ hence $X_4 = \{(0)\}$ and $\mathscr{O}_{X_4}(X_4) = \mathbb{C}$. ∎

[a] Note to self: The question of uniqueness of these functions in terms of their valuations are a separate question entirely. We just want to assign open sets of $\operatorname{Spec} R$ to subsets of R.

Problem 21

Let $\mathscr{U}$ be the set of open and dense sets in X. Compute the *ring of rational functions* $\varinjlim_{U \in \mathscr{U}} \mathscr{O}_X(U)$ first in the case where R is a domain and then for an arbitrary Noetherian ring.

Proof. First, recall (or learn for the first time) that if S is a multiplicative subset of a ring R, then $\varinjlim_{s \in S} R_s = S^{-1}R$. All primes of R correspond to irreducible components in X, but minimal primes correspond to special ones called an isolated components. Compare this to non-minimal primes, which are called embedded primes (geometrically, they are embedded in the isolated components).

> **Fact.** The complement of a dense open is a closed set $V(I)$, where I is not contained in any minimal prime of R.
>
> *Proof.* If I were contained in some minimal prime J, then $V(I)$ contains the irreducible component $V(J)$ of X. But that means that the complement of $V(I)$ (our dense open set) entirely misses an irreducible component, and is thus contained in the union of the remaining components. The closure of the "dense" open is thus bounded by the remaining components, so it cannot be dense. ■

I want to find f such that $\mathfrak{p} \in X_f$ and f does not belong to any minimal prime of R. Suppose that $\mathfrak{p} \in X - V(I)$, then $p \not\subset I$. The union of $\mathfrak{p}$ with every minimal prime also can't contain I, for if it did then prime avoidance implies that I would necessarily be a subset of either $\mathfrak{p}$ or a minimal prime. So there must be an $f \in I$ that is not in $\mathfrak{p}$ (hence $\mathfrak{p} \in X_f$ [b]) nor any minimal prime. Thus any dense open is covered by sets of the form X_f where f does not belong to any minimal prime of R (since $\mathfrak{p} \notin V(I)$.). So we can compute the direct limit with respect to R_f where $f \in S := \{f \in R \mid f \text{ not in any minimal prime of } R\}$. By the recalled fact on the first line, $\varinjlim_{f \in S} R_f = S^{-1}R$. Recall that the union of associated primes are the zero divisors of R, so S is precisely the set of non-zero divisors of R. This means that $S^{-1}R$ is the total ring of fractions of R. If R is a domain, then the total ring of fractions is the same thing as the field of quotients. ■

[b] If $\mathfrak{p} \in X_f$ then $\mathfrak{p} \notin V(f)$, which means that $\mathfrak{p}$ does not contain (f).