Geometry of Schemes: I.2 Schemes in General

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Problem 32

A scheme X is reduced if and only if every affine open subscheme of X is reduced, if and only if every local ring $\mathscr{O}_{X,p}$ is reduced for closed points $p \in X$.

Proof. If a scheme X is reduced, then the ring $\mathscr{O}_X(U)$ is reduced for every open set U. Suppose that U is an open affine subscheme – then $(U, \mathscr{O}_X|_U)$ is also a reduced scheme since every open set $V \subset U$ is open in X too, and thus $\mathscr{O}_X|_U(V) = \mathscr{O}_X(V)$ is a reduced ring. Conversely, suppose that every affine open subscheme of X is reduced but that some nonzero section $f \in \mathscr{O}_X(U)$ is nilpotent. Since our scheme is locally affine, U is covered by affine open subschemes U_i. We know that $f^n = 0$ is sent to 0 in each $\mathscr{O}_X(U_i)$ and since these rings are reduced, the inclusion of f in $\mathscr{O}_X(U_i)$ must be zero. But then by the gluing property, f must be zero in $\mathscr{O}_X(U)$ too, a contradiction.
For the second equivalence, suppose every affine open subscheme of X is reduced and pick any nonzero element $f \in \mathscr{O}_{X,p}$. Any representative $(U, f \in \mathscr{O}(U))$ of f is nonzero and hence not nilpotent. Hence f is not nilpotent in $\mathcal{O}_{X,p}$. Conversely, suppose that every local ring $\mathscr{O}_{X,p}$ is reduced for closed points $p \in X$ but that $f \in \mathscr{O}_X(U)$ is a section such that $f^n = 0$. Then the image of f in $\mathscr{O}_{U,u}$ is zero for all $u \in U$. Hence f is zero by the gluing axiom. ■

Problem 37

The underlying space of a zero-dimensional Noetherian scheme is finite.

Proof. Since a Noetherian scheme has a finite cover by open affine subschemes, we may reduce the statement to zero-dimensional affine schemes $X = \mathrm{Spec}(R)$ where R is Noetherian ring using the gluing property. Now, we just need to show that R has only finitely many prime ideals. This follows since a zero-dimensional Noetherian ring is the same as Artinian – all primes are maximal and there are only finitely many of them. (See atiyah an mcdonal chapter 8) ■