Geometry of Schemes: II.3-4 Reduced Schemes

John Cobb

Problem 11

Suppose that K is algebraically closed, and let $Z = \operatorname{Spec} K[x_1, \ldots, x_n]/I \subset \mathbb{A}^n_K$ be any subscheme of dimension 0 and degree 3, supported at the origin. Show that Z is isomorphic to either $X = \operatorname{Spec} K[x]/(x^3)$ or to

$$Y = \operatorname{Spec} K[x, y]/(x^2, xy, y^2)$$

and X, Y are not isomorphic to each other.

Proof. Since $Z = \operatorname{Spec} R$ is a subscheme of dimension 0, we must have a unique maximal ideal $\mathfrak{m}$ in R. Since R is three-dimensional, $\mathfrak{m}$ must be two-dimensional since $R/\mathfrak{m} \cong K$. We know for a fact that $\mathfrak{m}^3 = 0$. Suppose not – then $\mathfrak{m}^3 \subset \mathfrak{m}^2 \subset \mathfrak{m}$ would be a descending chain within the two-dimensional vector space $\mathfrak{m}$, so either $\mathfrak{m} = \mathfrak{m}^2$ (in which case Nakayama's lemma gives $\mathfrak{m} = 0$) or $\mathfrak{m}^2 = \mathfrak{m}^3$ (in which case Nakayama's lemma gives $\mathfrak{m}^2 = 0$). In either case, $\mathfrak{m}^3 = 0$. Let a and b generate $\mathfrak{m}$ and suppose first that $\mathfrak{m}^2 = 0$. Then the map $K[x, y] \to R$ sending $x \mapsto a$ and $y \mapsto b$ has $(x, y)^2 = (x^2, xy, y^2)$ in the kernel and identifies R with Y. Otherwise, it must be that $\mathfrak{m}^2 \neq 0$ but $\mathfrak{m}^3 = 0$. This implies that $\mathfrak{m}/\mathfrak{m}^2$ is a one-dimensional $R/\mathfrak{m} = K$-vector space. Nakayama's lemma says that $\mathfrak{m} = (a, b)$ must be one-dimensional too – in particular, there is some ring element r such that $a = rb$. This means that in the kernel of the surjection $K[x, y] \to R$ described above, we have $(x, y)^3$ and some element of the form $y - xf$ where f maps to r. Under the identification $y = xf$, $K[x, y] = K[x]$ and $(x, y)^3 = (x^3)$, identifying R with X. ■

Problem 15

Consider for $t \neq 0$ the subschemes

$$X_t = \{(0, 0), (t, 0), (0, t)\} \subset \mathbb{A}^2_K,$$

each consisting of three distinct points in $\mathbb{A}^2_K$.

Part A

Show that the limit scheme as $t \to 0$ is

$$X_0 = \operatorname{Spec} K[x, y]/(x^2, xy, y^2).$$

Proof. We define X_0 by taking its ideal to be the limit as $t \to 0$ of the ideal $I_t = (x, y) \cap (x - t, y) \cap (x, y - t)$. Now, since each of ideals in the intersection is comaximal so

$$\begin{aligned}
(x, y) \cap (x - t, y) \cap (x, y - t) &= (x^2 - tx, y^2, xy, yx - yt) \cap (x, y - t) \\
&= (x^2 - tx, y^2, xy, y) \cap (x, y - t) \\
&= (x^2 - tx, y) \cap (x, y - t) \\
&= (x^3 - tx^2, xy, yx^2 - txy - tx^2 + t^2 x, y^2 - yt) \\
&= (x^3 - tx^2, xy, x^2 + tx, y^2 - yt) \\
{}^{a} &= (x^2, xy, y^2 - yt)
\end{aligned}$$

Taking $t \to 0$,

$$I_0 = (x^2, xy, y^2).$$

■

Part B

Show that the restriction of a function $f \in K[x,y]$ on $\mathbb{A}^2_K$ on X_0 determines and is determined by the values at the origin of f and its first derivatives in every direction; thus we think of it as a first-order infinitesimal neighborhood of the point $(0,0)$.

Proof. A function f on X_0 takes its values in $K[x,y]/(x^2, xy, y^2)$. So f can be represented uniquely in the form

$$f(x,y) = a + bx + cy$$

since all higher terms are modded out. $f(x,y)$ is determined by

- its value at the origin, since $f(0,0) = a$,
- its derivative in the x-direction, since $\frac{\partial}{\partial x}(f) = b$,
- its derivative in the y-direction, since $\frac{\partial}{\partial y}(f) = c$.

Linear combinations of the derivative in the x and y directions give us first derivatives in every direction. ■

Part C

Show that X_0 is contained in the union of any two distinct lines through $(0,0)$.

Proof. Let X_1 and X_2 be schemes representing two distinct lines through $(0,0)$. That is,

$$X_1 = \operatorname{Spec} K[x,y]/(\alpha_1 x + \beta_1 y), \qquad X_2 = \operatorname{Spec} K[x,y]/(\alpha_2 x + \beta_2 y),$$

where $\alpha_1\beta_2 - \alpha_2\beta_1$ is not zero. The union of two schemes corresponds to the scheme associated to the ideal intersection. Thus

$$X_1 \cup X_2 = \operatorname{Spec} K[x,y]/(\alpha_1 x + \beta_1 y) \cap (\alpha_2 x + \beta_2 y).$$

To show that X_0 is contained in this, it suffices to show that

$$(\alpha_1 x + \beta_1 y) \cap (\alpha_2 x + \beta_2 y) \subset (x^2, xy, y^2).$$

The ideal on the left is radical since it is an intersection of prime ideals, so

$$(\alpha_1 x + \beta_1 y) \cap (\alpha_2 x + \beta_2 y) = \sqrt{(\alpha_1 x + \beta_1 y) \cap (\alpha_2 x + \beta_2 y))} = \sqrt{(\alpha_1\alpha_2 x^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)xy + \beta_1\beta_2 y^2)}$$

Computing the radical of a primary ideal is easy; for $(f) = (cf_1^{a_1} f_2^{a_2} \cdots f_n^{a_n})$, it is always true that $\sqrt{(f)} = (f_1 f_2 \cdots f_n)$ (see for example Ideals Varieties, and Algorithms page 186). Thus

$$\sqrt{(\alpha_1\alpha_2 x^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)xy + \beta_1\beta_2 y^2)} = (\alpha_1\alpha_2 x^2 + (\alpha_1\beta_2 + \alpha_2\beta_1)xy + \beta_1\beta_2 y^2).$$

This is clearly a subset of (x^2, xy, y^2).

■

Part D

Show that X_0 is not contained in any nonsingular curve and thus, in particular, is not the scheme-theoretic intersection of any two nonsingular curves in $\mathbb{A}^2_K$.

Proof. Suppose it were contained in some nonsingular curve. The nonsingular condition gives us that the zariski tangent space to X_0 at every point x has dimenson equal to $\dim(X_0, x)$. Note: If X is Noetherian, this occurs if and only if $\mathscr{O}_{X,x}$ is a regular local ring. This probably gives us some sort of problem. ■