Geometry of Schemes: II.3-4 Reduced Schemes

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Problem 33

Part A

Prove that a module M over a ring $R = K[t]_{(t)}$ is flat if and only if t is a nonzerodivisor on M, that is, if and only if M is torsion-free.

Proof. More generally, any module over a Dedekind domain is flat if and only if it is torsion-free – and we know that $K[t]_{(t)}$ is a Dedekind domain because localizations of Dedekind domains are discrete valuation rings (thanks Tejasi and Ivan). ■

Claim: Any module over a Dedekind domain is flat if and only if it is torsion-free.

Proof. It suffices to check the statement over $R_\mathfrak{m}$ for $\mathfrak{m} \subset R$ maximal. If $M_\mathfrak{m}$ is flat over $R_\mathfrak{m}$, than any $x \in R_\mathfrak{m}$ gives rise to an injective function $x : R_\mathfrak{m} \to R_\mathfrak{m}$ (since there are no zero-divisors) so by tensoring we get a map $x : M_\mathfrak{m} \to M_\mathfrak{m}$ that is injective by flatness. In other words, nonzero elements only annihilate the zero element, and $M_\mathfrak{m}$ is torsion-free which implies that M is torsion-free. [a] Still need other direction, which relies mainly on the fact that the localization of a dedekind domain is a discrete valuation ring. ■

Part B

Let $A = R[x_1, \dots, x_n]$ be a polynomial ring over $R = K[t]_{(t)}$, and let M be an A-module with free presentation

$$F_1 \xrightarrow{\phi} F_0 \longrightarrow M \longrightarrow 0.$$

Consider the module $\overline{M} := M/M_t$ over the factor ring $\overline{A} := A/tA$, and let

$$\overline{F}_1 \xrightarrow{\overline{\phi}} \overline{F}_0 \longrightarrow \overline{M} \longrightarrow 0$$

be the corresponding presentation. Show that M is flat over R if and only if every second syzygy of $\overline{M}$ over $\overline{A}$ can be lifted to a second syzygy over A in the sense that every element of the kernel of $\overline{\phi}$ comes from an element of the kernel of ϕ. (Something similar is true for any local base ring R with maximal ideal $\mathfrak{m}$ if M is finitely generated over A; this is a form of the "local criterion of flatness" - see, for example, Eisenbud [1995, Section 6.4] or Matsumura [1986, p. 174].

Proof. To be clear, a syzygy is a relation that module generators satisfy – its precisely an element in the image of ϕ. A second syzygy is a relation that generators of the first module of syzygies satisfy – that is, an element in the kernel of ϕ. Note that to get from the first free presentation to the next, we can tensor by A/tA. Similarly, breaking up the free resolution into short exact sequences, we see that

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{im}\phi \arrow[r, hook] & F_0 \arrow[r] \arrow[d, squiggly, "-\otimes A/tA"] & M \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{im}\overline{\phi} \arrow[r] & \overline{F}_0 \arrow[r] & \overline{M} \arrow[r] & 0
\end{tikzcd}$$

[a] This direction just needed integral domain.

Suppose M is flat, then tensoring preserves short exactness of this sequence. That is, the injection of the first module of syzygies $\operatorname{im}\phi$ into F_0 is still an injection after tensoring. $\operatorname{im}\phi$ is a submodule of a free module and is hence free, thus also flat. That means that the same considerations apply to

$$\begin{tikzcd}
0 \arrow[r] & \ker\phi \arrow[r, hook] & F_1 \arrow[r] \arrow[d, rightsquigarrow, "-\otimes A/tA"] & \operatorname{im}\phi \arrow[r] & 0 \\
0 \arrow[r] & \ker\overline{\phi} \arrow[r] & \overline{F}_1 \arrow[r] & \operatorname{im}\overline{\phi} \arrow[r] & 0
\end{tikzcd}$$

Since $\ker\overline{\phi}$ injects into $\overline{F}_1$, every element of the kernel of $\overline{\phi}$ comes from an element of the kernel of ϕ under the tensoring operation. The other direction follows the same argument in reverse. ■