Geometry of Schemes: II.4 Arithmetic Schemes

John Cobb

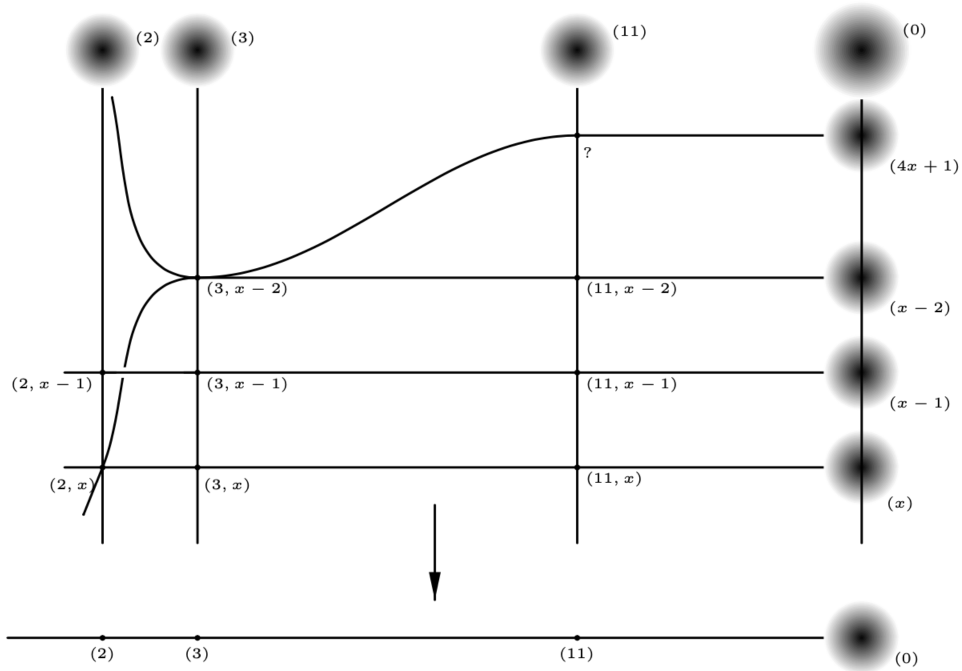

Figure 1: A depiction of a map $\operatorname{Spec} \mathbb{Z}[x] \to \operatorname{Spec} \mathbb{Z}$

Problem 38

What is the point marked with a ? in the picture above? Why are the closures of the points $(4x+1)$ and $(x-2)$ indicated by curves meeting tangentially at the point $(3, x-2)$ while they are both transverse to the closure of (3)? Why is the closure of the point $(4x+1)$ drawn as having a vertical asymptote over the point $(2) \in \operatorname{Spec} \mathbb{Z}$.

Proof. The points marked with a ? is $(11, 4x+1)$.
The tangentiality can be explained by the fact that the closure of the points $(4x+1)$ and $(x-2)$ do not meet at a reduced point. Their intersection is given by

$$\operatorname{Spec} \mathbb{Z}[x]/(4x+1) \cap \operatorname{Spec} \mathbb{Z}[x]/(x-2) = \operatorname{Spec} \mathbb{Z}[x]/(4x+1, x-2).$$

Using the isomorphism $x \mapsto 2$ we see that

$$\mathbb{Z}[x]/(4x+1, x-2) \cong \mathbb{Z}/9\mathbb{Z}.$$

So there is a double point at $(3, x-2)$. In contrast, the intersection

$$\operatorname{Spec} \mathbb{Z}[x]/(4x+1) \cap \operatorname{Spec} \mathbb{Z}[x]/(3) = \operatorname{Spec} \mathbb{Z}[x]/(4x+1, 3)$$

is a reduced ring, hence they meet at a reduced (nonsingular) point.
Lastly, lets compute the intersection of the closure of the point $(4x+1)$ and the fiber over (2). That is, we should compute the intersection of schemes of $\operatorname{Spec} \mathbb{Z}[x]/(4x+1)$ and $\operatorname{Spec} \mathbb{Z}[x]/(2)$. This is the same as $\operatorname{Spec} \mathbb{Z}[x]/(4x+1, 2)$. We can cancel the $4x$ using the 2, leaving $\operatorname{Spec} \mathbb{Z}[x]/(1, 2) = \operatorname{Spec} \mathbb{Z}[x]/(1) = \varnothing$. ■