Geometry of Schemes: III.1-III.2.3 Attributes of Morphisms

John Cobb

Problem 6

Part A

For any homogeneous ideal I of S and homogeneous element f of degree 1, the intersection

$$(I \cdot S[f^{-1}]) \cap S[f^{-1}]_0$$

is generated by elements obtained by choosing a set of homogeneous generators of I and multiplying them by the appropriate (negative) powers of f (see Exercise III-10 for the generalization where f has arbitrary degree). Thus the homogeneous primes of $S[f^{-1}]$ are in one-to-one correspondence with all the primes (no homogeneity condition) of the ring of elements of degree 0 in $S[f^{-1}]$; the correspondence is given by taking a homogenous prime $\mathfrak{p}$ of $S[f^{-1}]$ to $\mathfrak{q} = \mathfrak{p} \cap S[f^{-1}]_0$ and taking the prime $\mathfrak{q}$ of $S[f^{-1}]_0$ to $\mathfrak{q}S[f^{-1}]$.

Proof. Let φ be the map taking homogenous primes $\mathfrak{p}$ of $S[f^{-1}]$ to primes of $S[f^{-1}]_0$ via

$$\varphi : \mathfrak{p} \mapsto \mathfrak{q} \cap S[f^{-1}]_0.$$

Let ψ be the map in the other direction, taking primes $\mathfrak{q}$ in $S[f^{-1}]_0$ to homogenous primes in $S[f^{-1}]$ via

$$\psi : \mathfrak{q} \mapsto \mathfrak{q}S[f^{-1}].$$

I want to show that these maps are inverse and that they indeed respect the homogeneity conditions claimed (φ takes homogenous primes to primes and ψ takes primes to homogenous primes).
First, to show that $\varphi\psi = 1$, note that the zero-degree part of $\psi(\mathfrak{q}) = \mathfrak{q}S[f^{-1}]$ is exactly $\mathfrak{q}$ by definition.
Now to show $\psi\varphi = 1$, suppose $\mathfrak{p}$ is a homogeneous prime in $S[f^{-1}]$. Imagine s is an element in $\mathfrak{p}$ of degree d. Then since s/f^d is of degree 0, $s/f^d \in \varphi(\mathfrak{p})$. Then we can multiply s/f^d freely by f^d to get that $s \in \psi\varphi(\mathfrak{p})$. So $\mathfrak{p} \subseteq \psi\varphi(\mathfrak{p})$.
For the other direction, suppose $s \in \psi\varphi(\mathfrak{p})$ is again an element of degree d. As before, we know $s/f^d \in \varphi(\mathfrak{p})$ so there is some $s' \in \mathfrak{p}$ such that $s'/f^{d'} = s/f^d$. This implies that for some n, $f^n(f^d s' - f^{d'} s) = 0$. Since $f^{d+n}s' \in \mathfrak{p}$, it must be that $sf^{d'+n}$ is also in $\mathfrak{p}$. But $f^{d'+n} \notin \mathfrak{p}$ (because f is a unit!) so it must be that $s \in \mathfrak{p}$. Thus $\psi\varphi(\mathfrak{p} \subseteq \mathfrak{p}$. So ψ and φ are truly inverses to one another.
Of course, both maps preserve primeness because they are inverse maps of each other. The hard part of the proof verifies that ψ takes primes to homogenous primes – look at 4.5E Vakils notes. ∎

Part B

Let $S = A[x_0, \dots, x_r]$ be the polynomial ring, and let U be the open affine set $(\mathbb{P}^r_A)_{x_i}$, of $\mathbb{P}^r_A = \operatorname{Proj} S$. By definition

$$U = \operatorname{Spec} S[x_i^{-1}]_0.$$

Show that

$$S[x_i^{-1}]_0 = A[x'_0, \dots, x'_r]$$

the polynomial ring with generators $x'_j = x_j/x_i$. (Note that $x'_i = 1$, so that this is a polynomial ring in r variables.) Thus

$$(\mathbb{P}^r_A)_{x_i} = \mathbb{A}^r_A$$

so projective r-space has an open affine cover by $r+1$ copies of affine r-space, as described in Chapter I.

Proof. Since each $x'_j = x_j x_i^{-1}$ is inside $S[x_i^{-1}]_0$ (they are of degree 0), we know that $A[x'_0, \ldots, x'_r] \subseteq S[x_i^{-1}]_0$. I think this works: By definition,

$$S[x_i^{-1}] := A[x_1, \cdots, x_r, x_i^{-1}].$$

The grading here gives x_i^{-1} a degree of -1, and all others 1. The reason we can't just take A to be the 0 degree part now is because there can be cancellation of degrees. Multiplying every element by x_i^{-1} (which is just a change of basis) yields

$$A[x_1, \ldots, x_r, x_i^{-1}] \cong A[x'_1, \ldots, x'_r, x_i^{-1}]$$

Now, every element has 0 degree except x_i^{-1}, which has degree -1. So taking the zero degree is simple – just get rid of the stuff that is below degree 0 as follows

$$A[x'_1, \ldots, x'_r, x_i^{-1}]_0 = A[x'_1, \ldots, x'_r].$$

■

Part C

Consider the map $\alpha : S \to S[x_i^{-1}]_0$ obtained by mapping x_i to 1 and x_j to x'_j for $j \neq i$. Show from part (a) that if I is a homogeneous ideal of S, then

$$I' := I \cdot S[x_i^{-1}] \cap S[x_i^{-1}]_0 = \alpha(I)' \cdot S[x_i^{-1}]_0.$$

Proof. From part (a) we know that the intersection is generated by elements obtained by choosing a set of homogeneous generators of I and then multiplying them by the appropriate (negative) powers of x_i – this is precisely what the map $\alpha : S \to S[x_i^{-1}]$ does! ■