Geometry of Schemes: Week 4

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Problem 0.1. (II-2) We are given a ring homomorphism:

$$\varphi : K[x] \to K[x]$$

$$x \mapsto x^2$$

This map induces a map on schemes $\mathbb{A}^2_K \to \mathbb{A}^2_K$. Using the definition of a pre-image of a closed subscheme it is straightforward that the pre-image of (x) is that subscheme associated to the ideal $\varphi((x))K[x] = (x^2)$. That is, the subscheme $\operatorname{Spec} K[x]/(x^2)$. More generally, the prei-mage of $(x-a)$ is the subscheme $\operatorname{Spec} K[x]/(x^2 - a) \simeq \operatorname{Spec} K[x]/(x + \sqrt{a}) \times K[x]/(x - \sqrt{a}) \simeq K \times K$, (if $a \neq 0$ and is a square in K).

More explicitly, we can compute the pre-images of the prime ideals under the map

$$\varphi^* : \mathbb{A}^2_K \to \mathbb{A}^2_K$$

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$$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$$

We claim that $(x-a) \mapsto (x-a^2)$ under φ^*. Notice that $x^2 - a^2 \in (x-a)$ and so $(x-a^2) \subset \varphi^{-1}(x-a)$ but $(x-a^2)$ is maximal and so $\varphi^{-1}(x-a) = (x-a^2)$, that is $\varphi^*(x-a) = (x-a^2)$. Similarly, $\varphi^*(x+a) \mapsto (x-a^2)$.

This explains the picture in the book: Over (x), we just have one point (x) and over each $(x-a^2)$, we have two points $(x-a)$ and $(x+a)$.