Geometry of Schemes: Week 2

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1 Schemes in general

Problem 1.1 (I-24 part(a))**.** Given $Z = \operatorname{Spec}\mathbb{C}[x]$ and $X = Z/\{(x) \sim (x-1)\}$. Let $\varphi : Z \to X$ be the natural projection and $\mathcal{O}$ the pushward of the structure sheaf $\mathcal{O}_Z$.

A. We show that for all $f \in \mathcal{O}(X) = \mathbb{C}[x]$, the first condition in the definition of an affine scheme is not satisfied: Step 1 is to see at what the pullback sheaf looks like on X.

Case 1. Suppose $f(x) \in \mathbb{C}[x]$ such that x and $x-1$ do not divide $f(x)$: Then $\mathcal{F}(X_f)$[1] $= \mathcal{F}(\varphi^{-1}(X_f)) = \mathcal{F}(Z_f)$[2].

Case 2. Let $f(x)$ be such that $x \mid f$ but $x-1 \nmid f(x)$: In such a case $\mathcal{F}(\varphi^{-1}(X_f)) = \mathcal{F}(Z_f \backslash (x) \cup (x-1)) = \mathcal{F}(Z_{(x-1)f(x)})$.

Similarly, the same argument takes care of the case when $(x-1) \mid f(x)$ and $x \nmid f(x)$. We get that $\mathcal{F}(\varphi^{-1}(X_f)) = \mathcal{F}(Z_{xf(x)})$.

Case 3. The last case is when both x and $x-1$ divide f. It is easy to check that $\mathcal{F}(\varphi^{-1}(X_f)) = \mathcal{F}(Z_{f(x)})$.

Next step is to compute the stalks at any point in $|X|$. Let's do it for $\mathfrak{p} = (x-2) \in |X|$ and we will see that the computation generalizes for any $\mathfrak{p} = (x-a); a \neq 0, 1$. By definition $\mathcal{O}_{(x-2)} = \varinjlim_{(x-2)\in U} \mathcal{F}(U) = \varinjlim \mathbb{C}[x,y]_f$ as f varies all over $\mathbb{C}[x]$ with $(x-2) \nmid f$ and either $x(x-1) \mid f$ or x and $(x-1) \nmid f$. Note that in the directed system we will never have $\mathbb{C}[x]_x$ or $C[x]_{x-1}$ but we will always have $\mathbb{C}[x]_{x(x-1)}$, but this automatically contains

[1]The notation X_f is for basic open set in X consisting of prime ideals not containing f.

[2]Similary Z_f denotes the associated basic open set in Z

$\mathbb{C}[x]_x$ and $\mathbb{C}[x]_{x-1}$. Therefore, $\mathcal{O}_{x-2} = \mathbb{C}[x]_{(x-2)}$. In this case the second condition *is* satisfied.

Furthermore, $\mathcal{O}_{\overline{(x)}} = \mathcal{O}_{\overline{(x-1)}} = \varinjlim_{\overline{(x)} \in U} \mathcal{F}(U) = \varinjlim_{f \notin (x) \text{ or } (x-1)} \mathbb{C}[x]_f$ which results in inverting everything outside (x) or $(x-1)$. This is certainly not a local ring. Both (x) and $(x-1)$ will give us maximal ideals in this ring. Therefore, the first condition is *not* satisfied.

B. Next, we see in what cases is the first condition satisfied. We divide the computation again in 3 cases:

Case 1. Suppose $f(x) \in \mathbb{C}[x]$ such that x and $x-1$ do not divide $f(x)$: Then $f(x)$ is a unit in all $\mathcal{O}_{(x-a)}$ such that $(x-a) \nmid f$. So $U_f = \{(x-a) \nmid f\}$ and $\mathcal{F}(U_f) = \mathcal{F}(X_f) = \mathbb{C}[x][f^{-1}]$.

Case 2. Suppose $f(x) \in \mathbb{C}[x]$ such that $x \mid f$ but $x-1 \nmid f$: Then $f(x)$ is *not* unit in all $\mathcal{O}_{(x-a)}$ such that $(x-a) \mid f$. Note that it is also not a unit in $\mathcal{O}_{(x)} = \mathcal{O}_{(x-1)}$. So $U_f = \{(x-a) \nmid (x-a) \nmid f\}$ and $\mathcal{F}(U_f) = \mathcal{F}(X_f) = \mathcal{F}(Z_{(x-1)f(x)}) = \mathbb{C}[x]_{(x-1)f(x)} \neq \mathbb{C}[x][f^{-1}]$ and so the second condition is *not* met here. Same is the case when $x-1 \mid f(x)$ and $x \nmid f(x)$.

Case 3. Suppose $f(x) \in \mathbb{C}[x]$ such that x and $x-1$ both divide $f(x)$: Then $f(x)$ is not a unit in $\mathcal{O}_{(x)} = \mathcal{O}_{(x-1)}$ and in stalks at other prime ideals generated by factors of $f(x)$. But in this case the second condition is clearly met.

Problem 1.2 (I-24 part(b))**.** Given $Z = \mathbb{C}[x,y]; X = Z \backslash \{(x,y)\}$. We first show that $\mathcal{O}_Z|_X = \mathbb{C}[x,y]$. Since $\{x,y\} = V((x,y))$, we see that X is an open subset of Z and so can be covered by basic open sets. We can choose the basic open sets to be $Z_x \cup Z_y$. All we have to show is that on the intersection of these two open sets, if two sections agree then they have to be in $\mathbb{C}[x,y]$. Suppose $f(x,y)/x^n = g(x,y)/y^m$. Then $y^m f(x,y) = x^n g(x,y)$. This means that $f(x,y)$ does not have any purely y terms, since on the right hand side, we don't have any powers of y. Thus $x \mid f$ and so $f(x,y)/x^n = f'(x,y)/x^{n-1}$ for some polynomial $f'(x,y)$. Arguing similarly, with $f'(x,y)$, inductively, we see that $x^n \mid f(x,y)$ and so $f(x,y)/x^n \in \mathbb{C}[x,y]$.

Next, we would like to show that both the conditions 1 and 2 are satisfied in the definition of a scheme. This is because, we get the same stalks at all points in X : $\mathcal{O}_{\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in U} \mathcal{F}(U) = \varinjlim_{f \notin \mathfrak{p}} \mathcal{F}(Z_f \cap (Z_x \cup Z_y)) = \mathbb{C}[x,y]_{\mathfrak{p}}$. [3]

[3] The stalk at a point should not change if we shrink the neighborhoods around it.

The above computation tells us that the map $X \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$ is the inclusion map. More explicitly, let $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[x, y]$ be the map from the sheaf of functions on X to the stalk at a particular point $\mathfrak{p}$. The map between schemes is:

$$X \to |\operatorname{Spec} \mathbb{C}[x, y]|$$

$$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{m}_p)$$

where $\mathfrak{m}_p = \mathfrak{p}\mathbb{C}[x, y]_{\mathfrak{p}}$, the unique maximal ideal in $\mathbb{C}[x, y]_{\mathfrak{p}}$. But its pull back under φ is just $\mathfrak{p}$ and indeed we get the inclusion map.