Math 221 Worksheet 10
October 6, 2020
Section 2.8: Related Rates

1. A cube-shaped sponge is absorbing water, making it expand. Let $S(t)$ and $V(t)$ denote, respectively, the side-lengths and volume of the sponge at time t.

   (a) Find a function f such that $V(t) = f(S(t))$.

   $V(t) = S(t)^3$, so f is given by $f(x) = x^3$.

   (b) Describe the meaning of the derivatives $S'(t)$ and $V'(t)$. If we measure length in inches and time in minutes, what units do t, $S(t)$, $V(t)$, $S'(t)$, and $V'(t)$ have?

   $S'(t)$ – rate of change of the side-lengths as sponge expands
   $V'(t)$ – rate of change of volume " "

   units: t – min, $S(t)$ – in, $V(t)$ – in^3, $S'(t)$ – in/min, $V'(t)$ – in^3/min

   (c) What is the relation between $S'(t)$ and $V'(t)$?

   $V(t) = S(t)^3$, so $V'(t) = 3S(t)^2 S'(t)$.

   (d) When the volume of the sponge is 8 cubic inches, it is absorbing water at a rate of 2 cubic inches per minute. At that instant, how fast is the length of its sides changing?

   $S'(t) = \frac{V'(t)}{3S(t)^2} = \frac{V'(t)}{3V(t)^{2/3}}$. Plugging in $V(t) = 8$ and $V'(t) = 2$, we get $\boxed{S'(t) = \frac{2}{3 \cdot 8^{2/3}} = \frac{1}{6} \text{ in/min}}$

2. A snowball is melting, causing its surface area to decrease at a rate of 1 cm^2/min. How fast is its diameter decreasing at the moment when the diameter is 10 cm? (Recall that the surface area of a sphere of radius r is $4\pi r^2$.)

   Let D denote the diameter of the snowball and A its surface area. Then $A = \pi D^2$. So $A' = 2\pi D D'$. Plugging in $A' = -1$ and $D = 10$, we find that

   $$\boxed{D' = \frac{-1}{2\pi \cdot 10} = -\frac{1}{20\pi} \text{ cm/min.}}$$

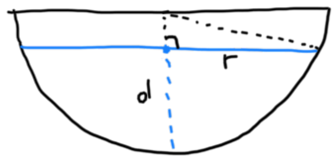
3. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is its surface area increasing at the moment when its sides have length 30 cm?

Let S = side length
A = surface area
V = volume.

Then $V = S^3$ and $A = 6S^2$. So $V' = 3S^2S'$ and $A' = 12SS'$. Together, these imply that $A' = \frac{4V'}{S}$. Plugging in $V' = 10$ and $S = 30$, we get

$$\boxed{A' = \frac{4 \cdot 10}{30} = \frac{4}{3} \text{ cm}^2/\text{min}}$$

4. A hemispherical bowl of radius 10 cm contains water to a depth of d cm. Find the radius r of the surface of the water as a function of d. Suppose that water is being added to the bowl, causing d to increase at a rate of 0.1 cm/hr. How fast is r increasing when $d = 5$ cm?



Pythagorean theorem: $r^2 + (10-d)^2 = 10^2$, so $r = \sqrt{10^2 - (10-d)^2}$.

$$r' = \frac{1}{2}\left(10^2 - (10-d)^2\right)^{-\frac{1}{2}}\left(-2(10-d)(-d')\right)$$
$$= (10-d)\left(10^2 - (10-d)^2\right)^{-\frac{1}{2}} d'.$$

Plugging in $d' = 0.1$ and $d = 5$, we get

$$\boxed{r' = (10-5)\left(10^2 - (10-5)^2\right)^{-\frac{1}{2}} \times 0.1 \approx 0.058 \text{ cm/hr.}}$$

5. A cylindrical swimming pool is being filled at a rate of 5 cubic feet per second. If the pool is 40 feet in diameter, how fast is the water level rising when the pool is one third full?

Let V = volume of water
H = water level (or height).

Since the pool has radius 20, we have $V = \pi \cdot 20^2 H = 400\pi H$.

So $V' = 400\pi H'$.

Plugging in $V' = 5$, we get $\boxed{H' = \frac{5}{400\pi} \approx 0.004 \ \frac{\text{ft}^3}{\text{sec}}}$

(Note that the water level rises at a constant rate, so we don't need to know the height at which the pool is one third full.)

6. The radius of a cylinder is increasing at a rate of 3 in./min., and its height is decreasing at a rate of 5 in./min. At what rate is the volume of the cylinder changing when its radius is 10 in. and its height is 15 in.? Is the volume increasing or decreasing?

Let r = radius, h = height, V = volume.

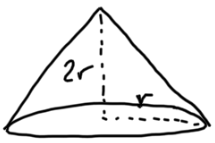
Then $V = \pi r^2 h$, so $V' = \pi(2rr'h + r^2 h')$.

Plugging in $r' = 3$, $h' = -5$, $r = 10$, and $h = 15$,

we get $\boxed{V' = \pi(2 \cdot 10 \cdot 3 \cdot 15 + 10^2(-5)) = 400\pi \text{ in}^3/\text{min.}}$

The volume is <u>increasing</u>.

7. Assume that sand slowly poured onto a level surface will pile in the shape of a cone whose height is equal to the diameter of its base. If sand is poured at 2 cubic meters per second, how fast is the height of the pile increasing when its base is 8 meters in diameter? (Recall that the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.)



Let V = volume $\qquad h = 2r$.

$= \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3$

Then $V' = \overset{2}{\cancel{16}}\pi r^2 r'$. Plugging in $V' = 2$ and $r = 4$, we find that $r' = \frac{1}{\underset{16}{\cancel{48}}\pi}$.

So the height (which is $2r$) is changing at a rate of

$\boxed{\frac{1}{\underset{8}{\cancel{24}}\pi} \text{ m}^3/\text{sec.}}$