Math 221 Worksheet 12
October 13, 2020
Sections 3.2 and 3.3: Mean Value Theorem and Things about Derivatives and Graphs

1. State the mean value theorem.

If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists a number $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

2. Let $f(x) = \cos(\pi x)\sqrt{2x+1}$. Show that there exists a number $c \in [0,4]$ such that $f'(c) = 1/2$.

By the mean value theorem, there exists a $c \in (0,4)$ such that

$$f'(c) = \frac{f(4)-f(0)}{4-0} = \frac{3-1}{4} = \frac{1}{2}.$$

3. Suppose that f is a differentiable function satisfying $f(1) = 10$ and $f'(x) \geq 2$ for every $x \in [1,4]$. Is it possible that $f(4) = 15$? Justify your answer.

No, it's not possible:

By the MVT, $\dfrac{f(4)-f(1)}{4-1} = f'(c)$ for some $c \in (1,4)$.

Since $f(1) = 10$ and $f'(c) \geq 2$, this implies that $f(4) \geq 16$.

4. Prove that $|\sin x| \leq |x|$ for all x.

It's true when $x = 0$. So assume $x \neq 0$. Then by the MVT,

$$|\sin x| = \frac{|\sin x - \sin(0)|}{|x-0|}|x| = |\cos(c)|\,|x|$$

for some c. Since $|\cos(c)| \leq 1$, this implies that $|\sin x| \leq |x|$.

5. Prove that the function $f(x) = x^3 + x + 1$ has exactly one (real) root. (Hint: Use the intermediate value theorem to show that f has a root, and use the mean value theorem to show that f does not have two roots.)

Since f is continuous and $f(-1) = -1$ and $f(0) = 1$, the IVT implies that f has a root.

If f had two roots — say $x = a$ and $x = b$ — then by the MVT,

$$0 = \frac{f(b)-f(a)}{b-a} = f'(c)$$

for some c. But $f'(x) = 3x^2+1$, so $f'(x)$ is never 0!

6. Suppose that f is a differentiable function satisfying $|f'(x)| < 1$ for every $x \in [0, 1]$. Prove that there exists at most one $c \in [0, 1]$ such that $f(c) = c$.

Let $g(x) = f(x) - x$. We want to show that g has at most one root in $[0,1]$. If it had two roots — say $x = a$ and $x = b$ — then by the MVT,

$$0 = \frac{g(b) - g(a)}{b-a} = g'(c)$$

for some c. But $g'(x) = f'(x) - 1$, so $g'(x)$ is never 0 since $|f'(x)| < 1$.

7. Let $f(x) = x^3 + x^2 - x + 1$. Determine where f is increasing and decreasing, and find its local minima and maxima.

First we solve $f'(x) = 0$:

$$f'(x) = 3x^2 + 2x - 1 = (x+1)(3x-1) \rightsquigarrow x = -1 \text{ or } x = \tfrac{1}{3}.$$

$f'(x) > 0$ when $x < -1$ or $x > \frac{1}{3}$ $\rightsquigarrow$ f increasing.

$f'(x) < 0$ when $-1 < x < \frac{1}{3}$ $\rightsquigarrow$ f decreasing.

f' switches from positive to negative at $x = -1$, so $f(-1) = 2$ is a local maximum.

f' switches from negative to positive at $x = \frac{1}{3}$, so $f(\frac{1}{3}) = \frac{22}{27}$ is a local minimum.

8. Let $f(x) = x^4 - 4x^3$.

(a) Determine where f is increasing and where is f decreasing.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3).$$

$f'(x) > 0$ when $x > 3$ $\rightsquigarrow$ f increasing

$f'(x) < 0$ when $x < 3$ $\rightsquigarrow$ f decreasing

(b) Find all local minima and maxima of f.

Since f' switches from negative to positive at $x = 3$, $f(3) = -27$ is a local minimum.

(The critical point $x = 0$ corresponds to neither a local min nor a local max, since f' does not change sign there.)

(c) Determine where f is concave up and where f is concave down.

$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

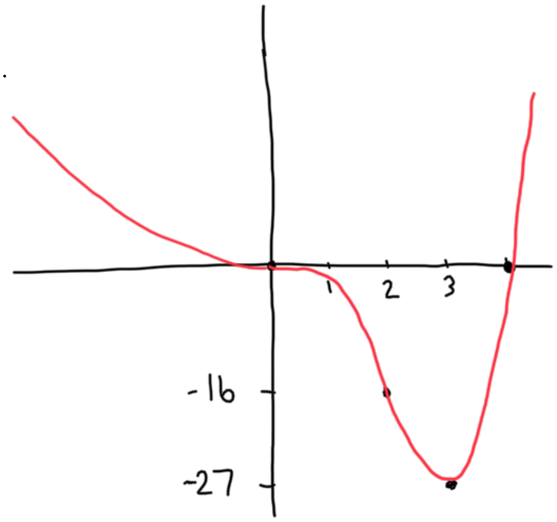
$f''(x) > 0$ when $x < 0$ or $x > 2$ $\rightsquigarrow$ f concave up

$f''(x) < 0$ when $0 < x < 2$ $\rightsquigarrow$ f concave down

(d) Find all inflection points of f.

f'' changes signs at $x=0$ and $x=2$, so $(0, f(0)) = (0,0)$ and $(2, f(2)) = (2,-16)$ are inflection points of f.

(e) Sketch the graph of f.



9. Repeat Problem 8 with f replaced by the function $g(x) = \sin(x) + \cos(x)$ defined for $x \in [0, 2\pi)$.

$g'(x) = \cos(x) - \sin(x)$

$g'(x) > 0$ when $0 < x < \frac{\pi}{4}$ or $\frac{5\pi}{4} < x < 2\pi$ $\rightsquigarrow$ g increasing.

$g'(x) < 0$ when $\frac{\pi}{4} < x < \frac{5\pi}{4}$ $\rightsquigarrow$ g decreasing.

$g\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum, since g' switches from positive to negative at $x = \frac{\pi}{4}$.

$g\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum, since g' switches from negative to positive at $x = \frac{5\pi}{4}$.

$g''(x) = -\sin(x) - \cos(x)$.

$g''(x) > 0$ when $\frac{3\pi}{4} < x < \frac{7\pi}{4}$ $\rightsquigarrow$ g concave up

$g''(x) < 0$ when $0 < x < \frac{3\pi}{4}$ or $\frac{7\pi}{4} < x < 2\pi$ $\rightsquigarrow$ g concave down

g'' switches signs at $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, so $\left(\frac{3\pi}{4}, g\left(\frac{3\pi}{4}\right)\right) = \left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, g\left(\frac{7\pi}{4}\right)\right) = \left(\frac{7\pi}{4}, 0\right)$ are inflection points of g.

See below for graph.

10. Show that $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n > 0$.

By the MVT, $\sqrt{n+1} - \sqrt{n} = \frac{1}{2\sqrt{c}}$ for some $c \in (n, n+1)$.

Thus,

$$\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}.$$

11. Show that if $f'(x) = 0$ for all x, then f is constant. If $f''(x) = 0$ for all x, what form must f take? What about if $f'''(x) = 0$ for all x?

If $f'(x) = 0$ for all x, then by the MVT $\frac{f(t) - f(0)}{t - 0} = 0$ for all t. This implies that $f(t) = f(0)$ for all t, so f is constant.

If $f''(x) = 0$ for every x, then f' is constant — say $f' = C$.
So $(f(x) - Cx)' = 0$ for all x, and thus $f(x) - Cx$ is a constant function — say $f(x) - Cx = C'$.
So $f(x) = Cx + C'$ (i.e. f is a linear function.)

If $f''' = 0$, then $f'(x) = Cx + C'$ for some C, C'.
So $\left(f(x) - \frac{C}{2}x^2 - C'x\right)' = 0$.
So $f(x) - \frac{C}{2}x^2 - C'x$ is constant. So f is quadratic! :-)

Graph for Problem 9