Math 221 Worksheet 12

October 13, 2020

Sections 3.2 and 3.3: Mean Value Theorem and Things about Derivatives and Graphs

1. State the mean value theorem.

2. Let $f(x) = \cos(\pi x)\sqrt{2x+1}$. Show that there exists a number $c \in [0, 4]$ such that $f'(c) = 1/2$.

3. Suppose that f is a differentiable function satisfying $f(1) = 10$ and $f'(x) \geq 2$ for every $x \in [1, 4]$. Is it possible that $f(4) = 15$? Justify your answer.

4. Prove that $|\sin x| \leq |x|$ for all x.

5. Prove that the function $f(x) = x^3 + x + 1$ has exactly one (real) root. (Hint: Use the intermediate value theorem to show that f has a root, and use the mean value theorem to show that f does not have two roots.)

6. Suppose that f is a differentiable function satisfying $|f'(x)| < 1$ for every $x \in [0, 1]$. Prove that there exists at most one $c \in [0, 1]$ such that $f(c) = c$.

7. Let $f(x) = x^3 + x^2 - x + 1$. Determine where f is increasing and decreasing, and find its local minima and maxima.

8. Let $f(x) = x^4 - 4x^3$.

   (a) Determine where f is increasing and where is f decreasing.

   (b) Find all local minima and maxima of f.

   (c) Determine where f is concave up and where f is concave down.

(d) Find all inflection points of f.

(e) Sketch the graph of f.

9. Repeat Problem 8 with f replaced by the function $g(x) = \sin(x) + \cos(x)$ defined for $x \in [0, 2\pi)$.

10. Show that $\frac{1}{2\sqrt{n+1}} < \sqrt{n+1} - \sqrt{n} < \frac{1}{2\sqrt{n}}$ for all $n > 0$.

11. Show that if $f'(x) = 0$ for all x, then f is constant. If $f''(x) = 0$ for all x, what form must f take? What about if $f'''(x) = 0$ for all x?