Math 221 - Week 7 - Worksheet 2

Topics: Section 3.4 - Limits at Infinity; Horizontal Asymptotes and Section 3.5 - Summary of Curve Sketching

Instructions: Listen to your TA's instructions. There are substantially more problems on this worksheet than we expect to be done in discussion, and your TA might not have you do problems in order. The worksheets are intentionally longer than will be covered in discussion in order to give students additional practice problems they may use to study. Do not worry if you do not finish the worksheet :).

1. Find all asymptotes of the function $f(x) = \dfrac{\sin(x)}{x}$.

VA : None $\quad \lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$

HA : $\lim_{x\to\pm\infty} \dfrac{\sin(x)}{x}$: $\quad -1 \le \sin(x) \le 1$

$$-\frac{1}{|x|} \le \frac{\sin(x)}{x} \le \frac{1}{|x|}$$

$$\boxed{y=0}$$

$$\lim_{x\to\pm\infty} -\frac{1}{|x|} = \lim_{x\to\pm\infty} \frac{1}{|x|} = 0$$

$$\Rightarrow \lim_{x\to\pm\infty} \frac{\sin(x)}{x} = 0 \quad \text{(By squeeze theorem)}$$

2. Let $g(x) = x^4 - 8x^2 + 1$. Determine the intervals on which the function is increasing and decreasing. Find the inflection points of g and intervals where g is concave up and concave down. Sketch the function.

$$g'(x) = 4x^3 - 16x = x(4x^2 - 16)$$

$$= 4x(x+2)(x-2)$$

Critical points : $x = 0, 2, -2$

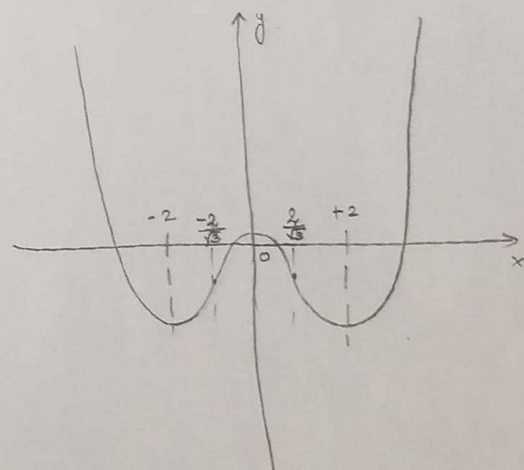
Slopes :

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
$g'(x)$	−	+	−	+
$g(x)$	↘	↗	↘	↗

$$g''(x) = 12x^2 - 16 = 12\left(x^2 - \tfrac{4}{3}\right) \Rightarrow x = \pm\frac{2}{\sqrt{3}} \quad \text{(Inflection pts)}$$

Curvature :

	$x<-2/\sqrt{3}$	$-2/\sqrt{3}<x<2/\sqrt{3}$	$x>2/\sqrt{3}$
$g''(x)$	+	−	+
$g(x)$	∪	∩	∪



3. Consider the function $f(x) = \dfrac{x^2 - x - 2}{x^2 - 1}$.

(a) Find all the asymptotes of the function.

HA :

$$\lim_{x \to \infty} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to \infty} \frac{\frac{x^2 - x - 2}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \to \infty} \frac{1 - \cancel{1/x}^{\,0} - \cancel{2/x^2}^{\,0}}{1 - \cancel{1/x^2}^{\,0}} = 1$$

$$\lim_{x \to -\infty} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to -\infty} \frac{\frac{x^2 - x - 2}{x^2}}{\frac{x^2 - 1}{x^2}} = \lim_{x \to -\infty} \frac{1 - \cancel{1/x}^{\,0} - \cancel{2/x^2}^{\,0}}{1 - \cancel{1/x^2}^{\,0}} = 1$$

$$\boxed{y = 1}$$

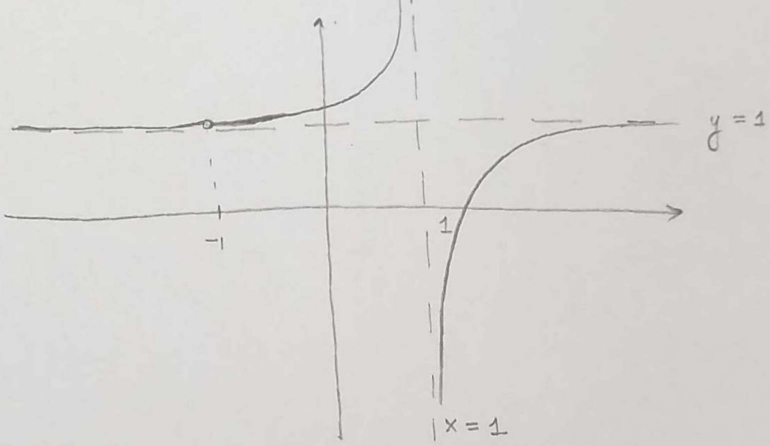
VA :

$$\lim_{x \to +1^+} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to 1^+} \frac{(x-2)^{\ominus}}{(x-1)^{\ominus}\,(\to 0^+)} = -\infty$$

$$\lim_{x \to +1^-} \frac{x^2 - x - 2}{x^2 - 1} = \lim_{x \to 1^-} \frac{(x-2)^{\ominus}}{(x-1)^{\ominus}\,(\to 0^-)} = +\infty$$

$$\boxed{x = 1}$$

(b) Sketch the graph of the function.



4. Find all the asymptotes of the following functions:

(a) $y = \dfrac{x^2}{x+1}$.

SA : (note that degree (x^2) (numerator) $= 1 +$ degree $(x+1)$ (denominator))

$$y = \frac{(x+1)(x-1) + 1}{x+1}$$

$$= (x-1) + \cancel{\frac{1}{(x+1)}} \to 0 \text{ (in the } \lim_{x \to \infty})$$

$$\begin{array}{r} x - 1 \\ x+1 \overline{)\, x^2 } \\ -\,x^2 + x \\ \hline -x \\ -(x-1) \\ \hline \underline{\underline{1}} \end{array}$$

$$\boxed{y = x - 1}$$

VA :

$$\lim_{x \to -1^+} \frac{x^2}{x+1} = \lim_{x \to -1^+} \frac{x^{2\,\oplus}}{x+1\,\oplus} = +\infty \qquad \lim_{x \to -1^-} \frac{x^2}{x+1} = \lim_{x \to -1^-} \frac{x^{2\,\oplus}}{x+1\,\ominus} = -\infty$$

$$\boxed{x = -1}$$

(b) $f(t) = -\frac{t^2-4}{t+1}$.

VA :

$$\lim_{t \to -1^+} -\frac{(t^2-4)^{\oplus}}{t+1_{(\to 0^+)}} = +\infty$$

$$\lim_{t \to -1^-} -\frac{(t^2-4)^{\oplus}}{t+1_{\ominus\,(\to 0^-)}} = -\infty$$

$\boxed{t = -1}$

SA :

$$f(t) = -\frac{t^2-4}{t+1} = -\frac{\cancel{t^2-1}^{\,(t-1)}}{\cancel{t+1}} - \frac{-3}{t+1}^{\to 0 \text{ (in the lim } t\to\pm\infty)}$$

$$\boxed{y(t) = -(t-1)}$$

(c) $g(z) = -\frac{z^2-z+1}{z-1}$.

VA : $\boxed{z = 1}$

$$\lim_{z \to 1^+} -\frac{(z^2-z+1)^{\ominus}}{(z-1)^{\oplus}} = -\infty$$

$$\lim_{z \to 1^-} -\frac{(z^2-z+1)^{\ominus}}{(z-1)\ \oplus} = +\infty$$

SA :

$$-\frac{(z^2-z)}{z-1} + \frac{-1}{z-1} = -z + \frac{-1}{z-1}^{\to 0 \text{ (in the lim } z\to\pm\infty)}$$

$$\boxed{y(z) = -z}$$

5. Consider the function $f(x) = \frac{x^2+4}{2x}$.

(a) Find all the asymptotes of the function.

VA : $\boxed{x = 0}$

$$\lim_{x \to 0^+} \frac{x^2+4}{2x} = +\infty$$

$$\lim_{x \to 0^-} \frac{x^2+4}{2x} = -\infty$$

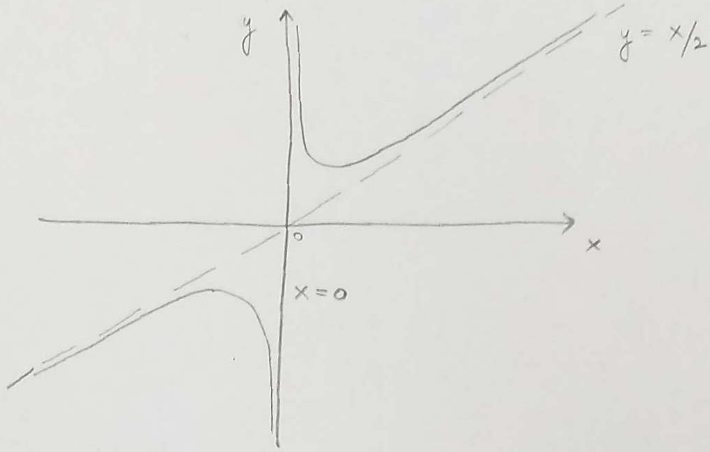
SA :

$$f(x) = \frac{x^2}{2x} + \frac{4}{2x} = x/2 + \underbrace{2/x}$$

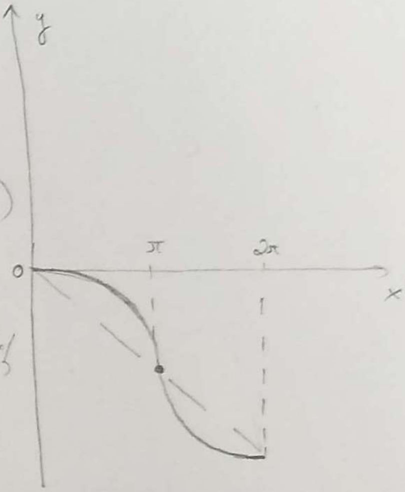
(this term goes to 0 as $x \to \pm\infty$)

$$\boxed{y = x/2}$$

(b) Sketch the function.



6. Sketch the graph of $f(x) = \sin(x) - x$ on the interval $[0, 2\pi]$.



A. Domain : $(-\infty, \infty)$. (Only need to show $f(x)$ in $[0, 2\pi]$)

B. Intercepts : x: $y = 0 \Rightarrow \sin(x) = x \Rightarrow x = 0$

C. Symmetry : $\sin(x) - x = -(\sin(-x) - (-x))$ (Doesn't matter for this problem interval $: [0, 2\pi]$)

$\Rightarrow$ Odd function.

D. No asymptotes

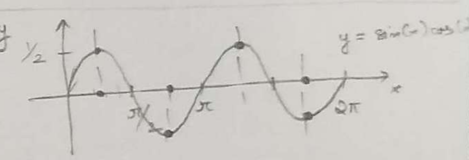
E. I/D : $f'(x) = \cos(x) - 1$

$\cos(x) - 1 = 0 \Rightarrow \cos(x) = 1 \Rightarrow x = 0, 2\pi$ (Both are not a local max/min)

F. Concavity : $f''(x) = -\sin(x)$. Inflection points : $x = 0, \pi, 2\pi$

7. Consider the function $f(x) = \sin(x)\cos(x)$ on the interval $[0, 2\pi]$.

(a) Sketch the graph of the function.



A. Domain : $(-\infty, \infty)$. Only need to plot $f(x)$ for x in $[0, 2\pi]$

B. Intercepts : x-intercept | Set $y = 0$, $\sin(x)\cos(x) = 0 \Rightarrow x = 0, \pi/2, \pi, 3\pi/2, 2\pi$

($\sin(x) = 0$ or $\cos(x) = 0$)

y-intercept | Set $x = 0$, $y = 0$.

C. Symmetry : (Antisymmetric / Odd) $\sin(x)\cos(x) = -(\sin(-x)\cos(-x))$

D. No asymptotes.

E. I/D : $f'(x) = -\sin^2(x) + \cos^2(x)$ (Drawtables)

Critical points : $f'(x) = 0 \Rightarrow \sin^2(x) = \cos^2(x)$

$\left[x = +\pi/4,\ \pi/2 + \pi/4,\ \pi + \pi/4,\ 3\pi/2 + \pi/4 \right]$ (in $[0, 2\pi]$)

Abs. max : $f(\pi/4) = 1/2$

Abs. min : $f(\pi/2 + \pi/4) = -1/2$

F. Concavity : $f''(x) = -4\sin(x)\cos(x)$

Inflection points : $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$

(b) Can you think of an easier method to sketch this function, without using Calculus (or a calculator)? (Hint: Try using a trig formula.)

$$f(x) = \sin(x)\cos(x) = \frac{1}{2}\left(2\sin(x)\cos(x)\right) = \frac{1}{2}\sin(2x)$$

Amplitude : $\frac{1}{2}$

Period : π.

8. Sketch the graph of $f(x) = x\sqrt{8 - x^2}$.

A. Domain : $[-2\sqrt{2}, 2\sqrt{2}]$ $\{$for $|x| > 2\sqrt{2}$, $8 - x^2 < 0\}$

B. Intercepts. $f(x) = 0$ when $x = 0 \Rightarrow$ passes through origin
$x = \pm 2\sqrt{2}$

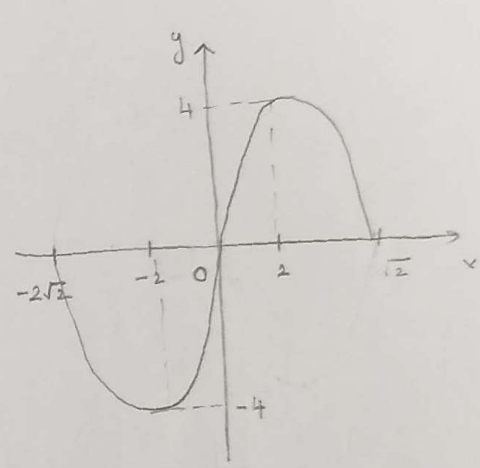
C. Symmetry : $x\sqrt{8-x^2} = -\left((-x)\sqrt{8-(-x)^2}\right)$

Odd function.

D. Asymptotes : None.

E. I/D. : $f'(x) = \dfrac{-2(x^2-4)}{\sqrt{8-x^2}}$; Critical points : $x = \pm 2$. (Draw table)

F. Concavity : $f''(x) = \dfrac{2x(x^2-12)}{(8-x^2)^{3/2}}$; Inflection points $x = \underbrace{\pm 2\sqrt{3}}_{\text{not in domain}}, 0$



9. Sketch the graph of $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$.

A. Domain : $(-\infty, \infty) - \{0\}$

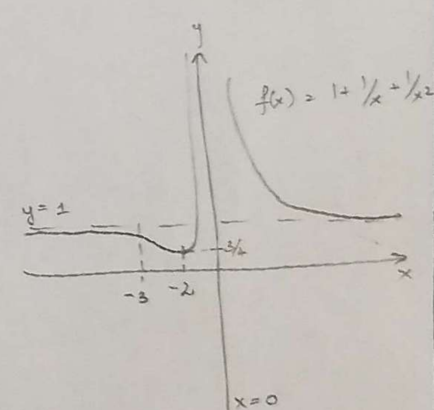
B. Intercepts : None

C. Symmetry : None

D. Asymptotes : VA : $\boxed{x = 0}$ $\lim_{x \to 0^+} f(x) = +\infty$.

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(1 + \frac{x+1}{x^2}\right) = +\infty$

HA : $\lim_{x \to \pm\infty} f(x) = 1 \Rightarrow \boxed{y = 1}$



E. $f'(x) = -\frac{1}{x^2} - \frac{2}{x^3} = \dfrac{-x-2}{x^3}$; $x = -2$ (local min) Critical point

	$x = -2$		$x = 0$	
$f'(x)$	−	+	−	
$f(x)$	↘	↗	↘	

F. $f''(x) = \dfrac{2(x+3)}{x^4}$; Inflection pt @ $x = -3$.

	$x = -3$		$x = 0$	
$f''(x)$	−	+	+	
$f(x)$	∩	∪	∪	

10. Show that the function $f(x) = \sqrt{4x^2+1}$ has two slant asymptotes: $y = 2x$ and $y = -2x$.

SA : $y = mx + c$

$$m = \lim_{x \to \pm\infty} \frac{f(x)}{x}$$

$$c = \lim_{x \to \pm\infty} f(x) - mx$$

① As $x \to \infty$

$$m = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt{4x^2+1}}{x} = \lim_{x\to\infty} \sqrt{\frac{4x^2}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \sqrt{4 + \cancel{1/x^2}^{\,0}} = \underline{\underline{2}}$$

$(x = \sqrt{x^2},$ when $x > 0)$

$$c = \lim_{x\to\infty} f(x) - mx = \lim_{x\to\infty} f(x) - 2x = \lim_{x\to\infty} \sqrt{4x^2+1} - 2x$$

$$\Rightarrow \lim_{x\to\infty} \frac{\left(\sqrt{4x^2+1} - 2x\right)\left(\sqrt{4x^2+1} + 2x\right)}{\left(\sqrt{4x^2+1} + 2x\right)} = \lim_{x\to\infty} \frac{\cancel{4x^2}+1 - \cancel{4x^2}}{\sqrt{4x^2+1} + 2x} = \underline{\underline{0}}$$

(Den $\to \infty$ as $x \to \infty$)

$\Rightarrow$ $\boxed{y = 2x}$

② As $x \to -\infty$

$$m = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x} = \lim_{x\to-\infty} -\sqrt{\frac{4x^2+1}{x^2}} = -\sqrt{4 + \cancel{1/x^2}^{\,0}}$$

$(x = -\sqrt{x^2},$ when $x < 0)$

$$= \underline{\underline{-2}}$$

$$c = \lim_{x\to-\infty} f(x) - mx = \lim_{x\to-\infty} \sqrt{4x^2+1} + 2x$$

$$= \lim_{x\to-\infty} \frac{\left(\sqrt{4x^2+1} + 2x\right)\left(\sqrt{4x^2+1} - 2x\right)}{\left(\sqrt{4x^2+1} - 2x\right)} = \lim_{x\to-\infty} \frac{(1) \div x}{\left(\sqrt{4x^2+1} - 2x\right) \div x}$$

$$= \lim_{x\to-\infty} \frac{\cancel{1/x}^{\,0}}{-\sqrt{4 + \cancel{1/x^2}^{\,0}} - 2} = 0/_{-4} = 0 /\!/$$

$\boxed{y = -2x}$