Math 221 - Week 8 - Worksheet 1
Topics: Section 3.7 - Optimization Problems

Instructions: Listen to your TA's instructions. There are substantially more problems on this worksheet than we expect to be done in discussion, and your TA might not have you do problems in order. The worksheets are intentionally longer than will be covered in discussion in order to give students additional practice problems they may use to study. Do not worry if you do not finish the worksheet :).

1. Find the absolute max of the following functions on the given intervals (if it exists). Justify why your answers are maxima and not minima.

   (a) $A(x) = 2x\sqrt{4-x^2}$, with $0 \le x \le 2$.

$$\frac{dA}{dx} = 2\sqrt{4-x^2} + 2x \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{4-x^2}}(-2x)$$

$$= \frac{2(4-x^2) - 2x^2}{\sqrt{4-x^2}} = \frac{4(2-x^2)}{\sqrt{4-x^2}}$$

; Critical points : $x = \pm\sqrt{2}$ ($-\sqrt{2}$ not in $[0,2]$)

	$x=-\sqrt{2}$		$x=\sqrt{2}$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

$\Rightarrow x = \sqrt{2}$ a local max

$f(\sqrt{2}) = 2\sqrt{2}\sqrt{4-2} = 4$, $f(0) = 0$ $\Rightarrow$ $(\sqrt{2}, 4)$ is the abs max

   (b) $P(x) = x(9-x)^2$, with $0 \le x \le 9$.

$$\frac{dP}{dx} = (9-x)^2 + 2x(9-x)(-1)$$

$$= (9-x)(9-x-2x)$$

$$= (9-x)3(3-x)$$

$\Rightarrow$ Critical pts : $x = 9, 3$.

	$x=3$		$x=9$	
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

$\Rightarrow (3, f(3))$ is a local max.

$f(3) = 3 \cdot 6^2 = 108$. $f(0) = f(9) = 0$

$\Rightarrow (3, 108)$ - abs max.

   (c) $V(x) = 12x - \frac{1}{4}x^3$, with $0 < x \le \sqrt{48}$.

$$\frac{dV}{dx} = 12 - \frac{3}{4}x^2$$

Critical pts : $12 - \frac{3}{4}x^2 = 0 \Rightarrow 16 - x^2 = 0 \Rightarrow x = \pm 4$.

	$x=-4$		$x=4$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

At $x=4$, $f(x)$ has a local max ($f(4) = 32$)

Also check end points $f(0) = 0$, $f(\sqrt{48}) < f(4)$ (from the table)

$\Big\}$ $(4, 32)$ is the abs. max.

2. Find the absolute min of the following functions on the given intervals (if it exists). Justify why your answers are minima and not maxima.

   (a) $f(r) = 2\pi r^2 + \frac{2000}{r}$ with $0 < r < \infty$.

$$\frac{df}{dr} = 4\pi r - \frac{2000}{r^2}$$

Set $4\pi r - \frac{2000}{r^2}$ to 0 to find critical pts

$f\left(\sqrt[3]{\frac{500}{\pi}}\right) = 300\sqrt[3]{2\pi}$

$$4\pi r - \frac{2000}{r^2} = 0 \Rightarrow r^3 = \frac{2000}{4\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

Second Derivative Test $\Big\}$ $f''(r) = 4\pi + \frac{4000}{r^3}$; $f''\left(\sqrt[3]{\frac{500}{\pi}}\right) > 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$ is a local min

(b) $c(x) = \dfrac{5000}{x} + 24x$, with $0 < x < \infty$.

Critical pts $(f'(x) = 0)$

$$\frac{dc}{dx} = \frac{-5000}{x^2} + 24 \,, \qquad \frac{5000}{x^2} = 24 \Rightarrow x = \pm\sqrt{\frac{5000}{24}}$$

(Note that $x = -\sqrt{\frac{5000}{24}}$ not in $0 < x < \infty$)

$$\frac{d^2c}{(dx)^2} = \frac{(5000)\cdot 2}{x^3} \,; \qquad \frac{d^2c}{(dx)^2}\left(\sqrt{\frac{5000}{24}}\right) > 0$$

$$\Rightarrow \left(\sqrt{\frac{5000}{24}},\ f\left(\sqrt{\frac{5000}{24}}\right)\right) \text{ is a local min.}$$

$= +400\sqrt{3}$

(c) $T(x) = \dfrac{1}{2}\sqrt{1+x^2} + \dfrac{1}{3} - \dfrac{1}{3}x$, with $0 \le x \le 1$.

Critical pts

$$\frac{dT}{dx} = \frac{1}{4}\,\frac{2x}{\sqrt{1+x^2}} - \frac{1}{3} \,, \qquad \frac{3x - 2\sqrt{1+x^2}}{6\sqrt{1+x^2}} = 0 \Rightarrow 3x = 2\sqrt{1+x^2}$$

$$9x^2 = 4(1+x^2)$$

$$5x^2 = +4$$

$$x = \pm\frac{2}{\sqrt{5}}$$

$x = \frac{2}{\sqrt{5}}$

$f'(x)$	$-$	$+$
	$\searrow$	$\nearrow$

$\Rightarrow \left(\frac{2}{\sqrt{5}}, T\left(\frac{2}{\sqrt{5}}\right)\right)$ is a local min.

$\left(\frac{2}{\sqrt{5}}, T\left(\frac{2}{\sqrt{5}}\right)\right) = \left(\frac{2}{\sqrt{5}}, \frac{1}{6}\left(2+\sqrt{5}\right)\right)$ is the abs. max ($T(x)$ is increasing as we move away from $x = \frac{2}{\sqrt{5}}$ (Table))

3. We will use calculus to find two numbers whose sum is 23 and whose product is a maximum.

(a) Write down any equations that you know (you'll need to come up with variables to represent the numbers we are looking for)

$$x + y = 23.$$

(b) Write down an equation that represents what we want to maximize.

$$F(x, y) = xy.$$

(c) Rewrite the above equation so that it is a function of a single variable.

$$f(x) = x(23 - x)$$

(d) Use calculus to find where the function is maximized.

$$f'(x) = 23 - 2x$$

$$23 - 2x = 0$$

$$x = 23/2$$

$$f''(x) = -2 \Rightarrow x = 23/2 \text{ is a maximum}$$

$$f(x) = \frac{23^2}{4}$$

(e) What are the two numbers that we want?

$$x, y = 23/2.$$

4. We want to find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

(a) Write down any equations that you know

$$x + y = 9 \quad , \quad x \geq 0 \quad y \geq 0.$$

(b) Write down an equation that represents what we want to maximize.

$$f(x, y) = xy^2$$

(c) Rewrite the above equation so that it is a function of a single variable.

$$f(x) = x(9-x)^2$$

(d) Use calculus to find where the function is maximized.

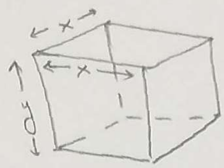
$$(3, f(3)) \Rightarrow (3, 108).$$ (Refer to 1b)

(e) What are the two numbers that we want? What is the maximum value achieved?

$$x = 3, \ y = 6.$$

5. If 1200 cm^2 of material is available to make a box with a square base and an open top, we want to find the largest possible volume of the box.

(a) Draw a picture (with variables) that represents the problem.



(b) Write down any equations that you know.

$$V(x,y) = x^2 y.$$

$$S(x,y) = 4xy + x^2 = 1200$$

(c) Write down an equation that represents what we want to maximize.

$$V(x,y) = x^2 y$$

(d) Rewrite the above equation so that it is a function of a single variable.

$$4xy + x^2 = 1200$$

$$y = \frac{1200 - x^2}{4x}$$

$$V(x) = x^2\left(\frac{1200 - x^2}{4x}\right) = \frac{\overset{300}{\cancel{1200}}\,x}{\cancel{4}} - x^3/4$$

(e) Use calculus to find where the function is maximized.

$$V'(x) = 300 - 3x^2/4 = 0$$

$$\Rightarrow x^2 = 400 \;;\; x = \pm 20$$

$x = 20$ cm (length has to be +ive).

Second-derivative test

$$V''(x) = -3x/2$$

$$V''(20) < 0$$

$\Rightarrow x = 20$ is a local max.

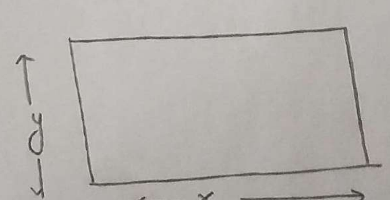
(f) What are the dimensions which maximize the volume? What is the volume achieved?

$x = 20$ cm $\quad y = 10$ cm.

$$V_{max} = V(20, 10) = 4000 \text{ cm}^3.$$

6. We show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

(a) Draw a picture that represents the problem.



(b) Let P denote the *given* perimeter. Keep in mind that P is *constant.* Write down any equations you know.

$$P = 2x + 2y = 2(x+y)$$

(c) Write down an equation that represents what we want to maximize.

$$A = xy$$

(d) Rewrite the above equation so that it is a function of a single variable.

$$A = x\left(P/2 - x\right)$$

(e) Use calculus to find where the function is maximized.

$$\frac{dA}{dx} = P/2 - 2x = 0$$

$$x = P/4 \quad ; \quad \frac{d^2A}{(dx)^2} = -2 \Rightarrow$$

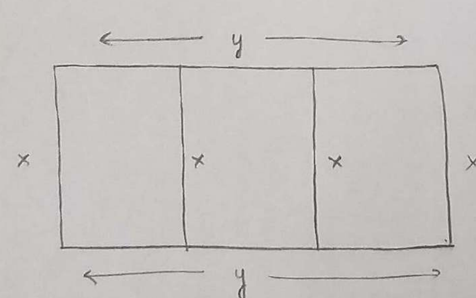
$x = P/4$ is a local max by the second derivative test

(f) Explain how this finishes the problem.

$$x = P/4, \quad y = P/2 - P/4 = P/4.$$

$$\Rightarrow x = y \Rightarrow \underline{\text{Square}}$$

7. Suppose we want to build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



Constraint: $4x + 2y = 500$.

$$A(x,y) = xy.$$

$$A(x, y(x)) = x/2\,(500 - 4x)$$

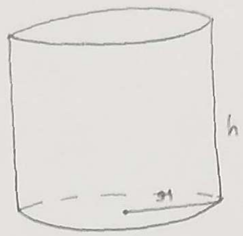
$$\frac{dA}{dx} = 250 - 4x = 0.$$

$$x = \frac{250}{4}\ \text{ft}$$

$$\frac{d^2A}{(dx)^2} = -4 \rightarrow$$ Second derivative test.

$$y = \underline{125\ \text{ft}}.$$

8. A container in the shape of a right circular cylinder with no top has surface area 3π square feet. What height h and base radius r will maximize the volume of the cylinder?



$$S = \pi r^2 + 2\pi r h = 3\pi \text{ sq. ft}$$

$$V = \pi r^2 h$$

$$h = \frac{3\pi - \pi r^2}{2\pi r}$$

$$V = \cancel{\pi} r^{\cancel{2}} \left(\frac{3\pi - \pi r^2}{2\cancel{\pi}\cancel{r}} \right)$$

$$= \frac{3\pi r - \pi r^3}{2}.$$

$$\frac{dV}{dr} = 3\pi/2 - 3\pi r^2/2 = 0$$

$$\pi - \pi r^2 = 0.$$

$$r = \pm 1 \quad (r = -1 \text{ is not physical})$$

$$\boxed{h = 1, r = 1}$$

9. You have been asked to design a one liter can shaped like a right circular cylinder. What dimensions will use the least material?

$$V = 1L = 1000 \text{ cm}^3. = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}.$$

Minimize $\rightarrow$ $S = 2\pi r^2 + 2\pi r h.$

$$S = 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2} \right)$$

$$= 2\pi r^2 + \frac{2000}{r}.$$

$\Rightarrow$ Abs. min $\Rightarrow$ $r = \sqrt[3]{\frac{500}{\pi}}$, $h = \frac{1000}{(500\pi)^{2/3}}$. (Refer to Q2)

10. What is the smallest possible perimeter for a rectangle with area 50 m^2?

Minimize $\rightarrow$ $P = x + y$

Constraint. $\rightarrow$ $A = xy = 50.$

$$y = \frac{50}{x}.$$

$$p(x) = x + \frac{50}{x}$$

$$\frac{dp}{dx} = 1 - \frac{50}{x^2} = 0.$$

Second derivative test

$$\left(\frac{d^2p}{dx^2} = + \frac{50}{x^3} \right)$$

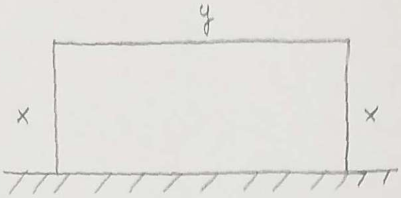
$$x^2 = 50$$

$$x = \pm 5\sqrt{2}$$

$x = +5\sqrt{2}$ m (-ive lengths are not physical!)

$$y = 5\sqrt{2}. \text{ m}.$$

11. You need to enclose a rectangular field with a fence. You have 500 feet of fencing material and there is a building on one side, so that side doesn't need any fencing. What is the largest possible area you can enclose?



$2x + y = 500$

$A = xy$

$A = x(500 - 2x)$

$\frac{dA}{dx} = 500 - 4x = 0$

$x = \frac{500}{4}$

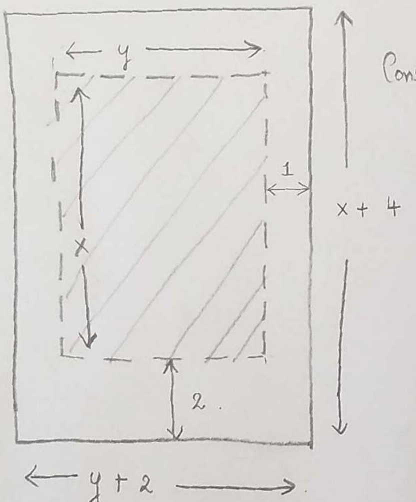
$x = 125$ ft

Second derivative test : $\frac{d^2A}{(dx)^2} = -4$

$\Rightarrow x = 125$ ft is maximum.

$A = 125 * (250)$ ft².

12. A printer needs to create a poster which will have a total area of 200 in^2 and will have 1 inch margins on the sides and 2 inch margins on the top and bottom. What dimensions will give the largest printed area? (The margins don't count as printed area).



Constraint : $(x+4)(y+2) = 200$

$y + 2 = \frac{200}{x+4}$

$= \frac{200}{x+4} - 2$

Maximize : $A(x, y) = xy$

$A(x, y(x)) = x\left(\frac{200}{x+4} - 2\right)$

$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{200x}{x+4} - 2x\right)$

$= \frac{200(x+4) - 200x}{(x+4)^2} - 2$

$= \frac{800}{(x+4)^2} - 2$

$800 = 2(x+4)^2$

$400 = (x+4)^2$

$x = (\pm 20) - 4$

$x = 20 - 4 = 16$ in (dimensions cannot be -ive)

$y = \frac{200}{16+4} = 10$ in

$\frac{d^2A}{(dx)^2} = \frac{-2 \cdot 800}{(x+4)^3} < 0$ for $x > 0$

$\Rightarrow \boxed{x = 16,\ y = 10}$ is local maximum.