Math 221 Worksheet 17
October 29, 2020
Section 4.1: Area and Riemann Sums; Section 3.9: Antiderivatives

1. Evaluate the following sums:

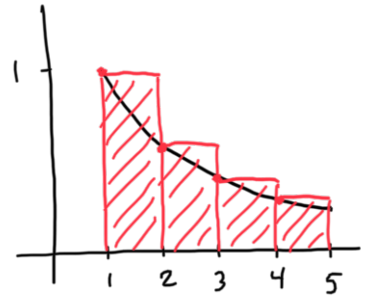
   (a) $\sum_{i=0}^{3} 2^i$

   (b) $\sum_{n=1}^{4} n^2$

   (c) $\sum_{j=10}^{100} (-1)^j$

2. (a) Estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$ by forming a Riemann sum of four rectangles using the right endpoints. Is your estimate too high or too low?

(b) Repeat part (a) using the left endpoints.



$$\sum_{i=1}^{4} \frac{1}{1+(i-1)\frac{5-1}{4}} \cdot \frac{5-1}{4} = \sum_{i=1}^{4} \frac{1}{i}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$= \frac{25}{12}.$$

This is an overestimate.

3. (a) Write the area under the graph of $f(x) = x^3$ from $x = 0$ to $x = 3$ as a limit of Riemann sums. (You do not need to evaluate the limit.)

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(0 + i \cdot \frac{3-0}{n}\right)^3 \cdot \frac{3-0}{n}$$

(These Riemann sums used right endpoints. You could also use left endpoints, etc.)

(b) Write the area under the graph of $f(x) = \frac{2x}{x^2+1}$ from $x = 1$ to $x = 3$ as a limit of Riemann sums. (You do not need to evaluate the limit.)

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2\left(1 + i \cdot \frac{3-1}{n}\right)}{\left(1 + i \cdot \frac{3-1}{n}\right)^2 + 1} \cdot \frac{3-1}{n}$$

(Same comment as above)

4. (a) Using the fact that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(2n+1)(n+1)n}{6}$, evaluate $\lim_{n\to\infty} \sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n}$.

$$\sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n} = \sum_{j=1}^{n} j^2 \cdot \frac{8}{n^3}$$

$$= \frac{8}{n^3} \sum_{j=1}^{n} j^2 = \frac{8}{n^3} \cdot \frac{(2n+1)(n+1)n}{6}.$$

So
$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(j \cdot \frac{2}{n}\right)^2 \frac{2}{n} = \lim_{n\to\infty} \frac{8(2n+1)(n+1)n}{6n^3} = \frac{16}{6} = \frac{8}{3}.$$

(b) Explain why the limit from part (a) is equal to the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 2$.

5. Use the idea of area to evaluate $\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \sqrt{1 - \left(\frac{j}{n}\right)^2}$. (Hint: Think about the function $f(x) = \sqrt{1 - x^2}$.)

6. A stone is dropped off a cliff and hits the ground at a speed of 120 feet per second. Assuming the acceleration due to gravity is 32 feet per second, what is the height of the cliff?