Math 221 Worksheet 18

November 3, 2020

Section 4.2: The Definite Integral

1. Find the area under the graph of $f(x) = x^2$ from $x = 0$ to $x = 3$ by evaluating a limit of Riemann sums.

The area is $\lim_{n \to \infty} \sum_{i=1}^{n} \left(i \cdot \frac{3}{n}\right)^2 \cdot \frac{3}{n}$. (Here, right endpoints were used, but other "sample points" would give the same limit.)

$$\text{So area} = \lim_{n \to \infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 \quad \left(\text{recall: } 1^2 + 2^2 + \cdots + n^2 = \frac{(2n+1)(n+1)n}{6}\right)$$

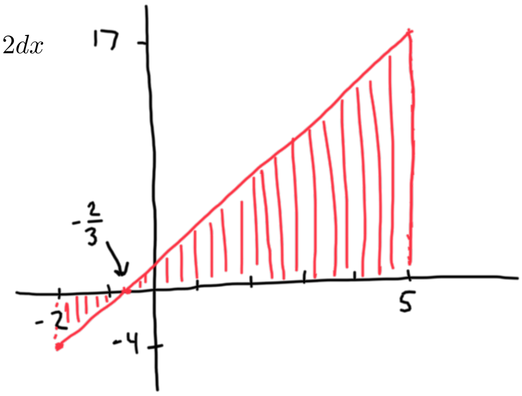
$$= \lim_{n \to \infty} \frac{27}{n^3} \cdot \frac{(2n+1)(n+1)n}{6}$$

$$= \frac{54}{6}$$

$$= 9.$$

2. Evaluate the following definite integrals:

(a) $\int_{-2}^{5} 3x + 2dx$

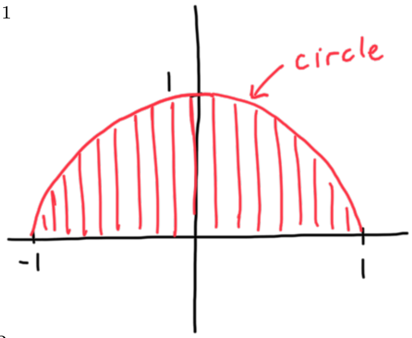


$$\int_{-2}^{5} (3x+2)\,dx$$

$$= \frac{1}{2} \cdot \frac{17}{3} \cdot 17 - \frac{1}{2} \cdot \frac{4}{3} \cdot 4$$

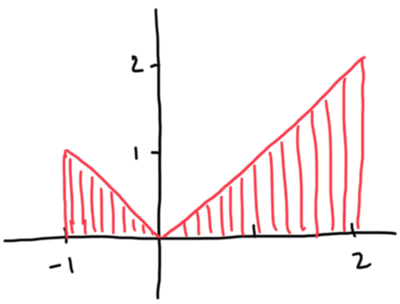
$$= \frac{91}{2}.$$

(b) $\int_{-1}^{1} \sqrt{1 - x^2} dx.$



$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\pi.$$

(c) $\int_{-1}^{2} |x| dx$



$$\int_{-1}^{2} |x|\,dx = \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 2$$

$$= \frac{5}{2}.$$

3. Suppose f and g are continuous functions on $[0,4]$ satisfying $\int_0^1 f(x)dx = 4$, $\int_0^4 f(x)dx = -6$, $\int_0^1 g(x)dx = -2$, and $\int_1^4 g(x)dx = 13$. Determine the following:

   (a) $\int_1^4 f(x)dx + \int_1^1 g(x)dx$

   (b) $\int_0^4 f(x) - g(x)dx$

   (c) $\int_4^1 2f(x) + 3g(x)dx$

4. Explain why $2 \leq \int_{-1}^1 \sqrt{1+x^2}\, dx \leq 2\sqrt{2}$.

5. Estimate the following definite integrals. (Hint: first determine the maximum and minimum values of the integrand on the interval over which you're integrating.)

   (a) $\int_1^3 x^2 dx$

   (b) $\int_4^9 (\sqrt{x} + x)dx$

(c) $\int_{32}^{64} \log_2(x)dx$

(d) $\int_0^{1/2} \frac{1}{\sin(\pi x)+4}dx$

6. Let $f(x) = 1 + \sqrt{9 - x^2}$.

   (a) Sketch the graph of f on the interval $[-3, 0]$. What is the area under the graph on this interval?

   (b) What is $\displaystyle\int_{-3}^{0} (1 + \sqrt{9 - x^2})dx$?

7. Let $f(x) = 2x$.

   (a) Sketch the graph of f, and label a point z on the positive x-axis.

   (b) Let $F(z)$ be the area under the graph of f on the interval $[0, z]$. Determine $F(z)$.

(c) How does $F(x)$ relate to $f(x)$?

(d) Use area to determine $\int_{-2}^{1} f(x)dx$.

(e) Calculate $F(1) - F(-2)$, and compare this to the integral from part (d). What's going on