Math 221 Worksheet 19
Section 4.3: The Fundamental Theorem of Calculus

1. State the fundamental theorem of calculus.

2. Use the fundamental theorem of calculus to evaluate $\int_0^3 x^2 dx$. Compare this to Problem 1 from Worksheet 18.

3. Use the fundamental theorem of calculus to determine the following:

   (a) $\frac{d}{dx}\left(\int_0^x \sqrt{1-t^2}dt\right)$

   (b) $\frac{d}{dx}\left(\int_x^{-5} t^3 - 2t^2 + 1\ dt\right)$

   (c) $\frac{d}{dx}\left(\int_2^{7x+3} t^2 dt\right)$

   (d) $\frac{d}{dx}\left(\int_2^{1/x} \arctan t\ dt\right)$

4. Let $F(x) = \int_2^x \frac{1}{1+t+t^2} dt$. Determine the region on which F is concave up.

$$F'(x) = \frac{1}{1+x+x^2} \text{ (by FTC), so } F''(x) = -\frac{1+2x}{(1+x+x^2)^2}.$$

So $F''(x) > 0$ when $x < -\frac{1}{2}$.

5. Use the fundamental theorem of calculus to evaluate the following:

(a) $\int_1^4 (2x^4 - 3x^2)dx$

$$= \left(\frac{2}{5}x^5 - x^3\right)\Big|_1^4 = \frac{2}{5}4^5 - 4^3 - \left(\frac{2}{5}\cdot 1^5 - 1^3\right)$$

(b) $\int_0^4 x\sqrt{x^3}dx$

$$= \int_0^4 x^{5/2} dx = \frac{2}{7}x^{7/2}\Big|_0^4 = \frac{2}{7}\cdot 4^{7/2}$$

(c) $\int_0^{\frac{\pi}{4}} \sin(x)dx$

$$= -\cos(x)\Big|_0^{\pi/4} = -\frac{1}{\sqrt{2}} + 1$$

(d) $\int_0^1 (x^3 - 1)^2 dx$

$$= \int_0^1 (x^6 - 2x^3 + 1)\,dx = \left(\frac{1}{7}x^7 - \frac{1}{2}x^4 + x\right)\Big|_0^1$$

$$= \frac{1}{7} - \frac{1}{2} + 1$$

6. Compute $\int_{-1}^1 (x + x^3)dx$. Given that you integrated an *odd* function, is there a geometric explanation for your answer?

$$\int_{-1}^1 (x+x^3)\,dx = \left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_{-1}^1 = \frac{1}{2} + \frac{1}{4} - \left(\frac{1}{2} + \frac{1}{4}\right) = 0.$$

Since $x + x^3$ is odd, the "positive area" under its graph on $[0,1]$ is exactly canceled by "negative area" on $[-1,1]$. (Sketch the graph!)

7. Let f be a continuous function satisfying $\int_1^5 f(t)dt = 8$.

(a) Let $F(x) = \int_0^x f(t)dt$. Show that $\frac{F(5)-F(1)}{5-1} = 2$.

(b) Prove that there exists $x \in (1,5)$ such that $f(x) = 2$.

8. Let $f(x) = \frac{1}{3}x$ and $g(x) = \sqrt{x}$.

(a) Find all points at which the graphs of f and g intersect.

(b) Find the area of the bounded region enclosed by the graphs of f and g.

9. (**Fun/optional**) Let f be a continuous function and let c be a real number. Prove that

$$\lim_{r \to 0^+} \frac{1}{2r} \int_{c-r}^{c+r} f(x)dx = f(c).$$