Math 221 Worksheet 21
November 12, 2020
Section 6.1: Inverse Functions; Section 6.2: Exponential Functions and their Derivatives

1. Compute the derivatives of the following functions:

(a) $f(x) = \dfrac{e^{4x}}{5x}$

$$f'(x) = \frac{5x \cdot e^{4x} \cdot 4 - e^{4x} \cdot 5}{25x^2}$$

(b) $f(x) = e^{x^2+5x}$

$$f'(x) = e^{x^2+5x}(2x+5)$$

(c) $f(x) = e^{2x}\sin(x)$

$$f'(x) = e^{2x} \cdot 2 \cdot \sin(x) + e^{2x}\cos(x)$$

(d) $f(x) = \sin(e^{2x})$

$$f'(x) = \cos(e^{2x})e^{2x} \cdot 2$$

(e) $F(x) = \displaystyle\int_2^x e^{\cos(\tan(t^2))}dt.$

$$F'(x) = e^{\cos(\tan(x^2))}$$ (fundamental theorem of calculus)

2. Evaluate the following definite integrals:

(a) $\displaystyle\int_{-1}^{2} (x^5 + e^x)\, dx$

$$= \left(\frac{1}{6}x^6 + e^x\right)\Big|_{-1}^{2}$$

$$= \frac{1}{6} \cdot 2^6 + e^2 - \left(\frac{1}{6}(-1)^6 + e^{-1}\right)$$

(b) $\displaystyle\int_0^{1/2} (e^y + 2\cos(\pi y))\, dy$

$$= \left(e^y + \frac{2}{\pi}\sin(\pi y)\right)\Big|_0^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} + \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) - \left(e^0 + \frac{2}{\pi}\sin(0)\right)$$

$$= e^{\frac{1}{2}} + \frac{2}{\pi} - 1$$

(c) $\displaystyle\int_1^2 \frac{e^{\sqrt{2x}}}{\sqrt{x}}\,dx$

(d) $\displaystyle\int_0^\pi \frac{\cos(x)}{e^{\sin^2(x)}}\,dx$

3. For each of the following functions, determine its domain and range and whether it is one-to-one. If it is one-to-one, find its inverse.

   (a) $f(x) = 4x - 5$

   (b) $f(x) = x^2 - 5x$.

   (c) $f(x) = \sin(2x)$.

   (d) $f(x) = \dfrac{3x-1}{2x+1}$

4. Compute the following limits:

(a) $\lim_{x\to\infty} \frac{e^{4x} - e^{-4x}}{2e^{4x} + e^{-4x}}$

$$= \lim_{x\to\infty} \frac{e^{4x} - e^{-4x}}{2e^{4x} + e^{-4x}} \cdot \frac{e^{-4x}}{e^{-4x}} = \lim_{x\to\infty} \frac{1 - e^{-8x}}{2 + e^{-8x}} = \frac{1}{2}$$

(b) $\lim_{x\to\infty} e^{-x}\sin(3x^2)$

$-e^{-x} \le e^{-x}\sin(3x^2) \le e^{-x}$ and

$\lim_{x\to\infty} e^{-x} = 0 = \lim_{x\to\infty} -e^{-x}$, so

$\lim_{x\to\infty} e^{-x}\sin(3x^2) = 0$ by the squeeze theorem.

5. If f is an invertible function and g is its inverse, then

$$g'(f(x)) = \frac{1}{f'(x)},$$

provided $f'(x) \neq 0$.

(a) Use implicit differentiation to prove the above formula.

Let $y = f(x)$. Then $g(y) = x$. So $g'(y)\,y' = 1$,

and thus

$$g'(f(x)) = g'(y) = \frac{1}{y'} = \frac{1}{f'(x)}.$$

(b) Given that the natural logarithm function $\ln x$ is the inverse of the natural exponential function e^x, what is $(\ln x)'$?

Let $x = e^t$. Then

$$(\ln x)' = \frac{1}{\frac{d}{dt}e^t} = \frac{1}{e^t} = \frac{1}{x}.$$

(c) Evaluate $\int_1^2 \frac{1}{x}\,dx$.

$$\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln 2.$$