Math 221 - Week 12 - Worksheet 1

Topics: Section 6.3 - Logarithmic Functions and Section 6.4 - Derivatives of Logarithmic Functions

Instructions: Listen to your TA's instructions. There are substantially more problems on this worksheet than we expect to be done in discussion, and your TA might not have you do problems in order. The worksheets are intentionally longer than will be covered in discussion in order to give students additional practice problems they may use to study. Do not worry if you do not finish the worksheet :).

1. Compute the derivatives of the following functions.

   (a) $f(x) = \ln(3x^2 - 5x)$ Chain rule.

   $$f'(x) = \frac{1}{3x^2-5x} \cdot (3x^2-5x)' = \frac{6x-5}{3x^2-5x}$$

   (b) $g(u) = \dfrac{u + \ln(5u)}{\sin(u)}$.   Quotient rule.

   $$g'(u) = \frac{(1+\frac{5}{5u})\sin u - \cos u\,(u+\ln(5u))}{\sin^2(u)}.$$

   (c) $f(s) = \ln\left(\sqrt{\frac{2s+1}{4s}}\right)$. $= \frac{1}{2}\ln\left(\frac{2s+1}{4s}\right) = \frac{1}{2}\left(\ln(2s+1) - \ln(4s)\right)$.

   $$f'(s) = \frac{1}{2}\left(\frac{2}{2s+1} - \frac{4}{4s}\right).$$

   (d) $h(u) = e^{4u}\ln(ue^u)$ $= e^{4u}(\ln u + u)$.   Product rule.

   $$h'(u) = 4e^{4u}(\ln u + u) + e^{4u}\left(\frac{1}{u} + 1\right).$$

   (e) $y = x\log_4(\sin(x))$ $= x\,\dfrac{\ln(\sin x)}{\ln 4}$.

   $$y' = \frac{\ln(\sin x)}{\ln 4} + \frac{x}{\ln 4}(\ln(\sin x))' = \frac{\ln(\sin x)}{\ln 4} + \frac{x}{\ln 4}\cdot\frac{\cos x}{\sin x}$$

   (f) $y = \log_2(x\log_5 x)$

   $$= \frac{\ln x + \ln(\ln x) - \ln\ln 5}{\ln 2}$$

   $$y' = \frac{1}{x\ln 2} + \frac{1}{\ln 2}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}.$$

2. Find the equation of the tangent line to the curve $y = \ln(x^2)$ at the point $(e, 2)$.

$= 2\ln x.$

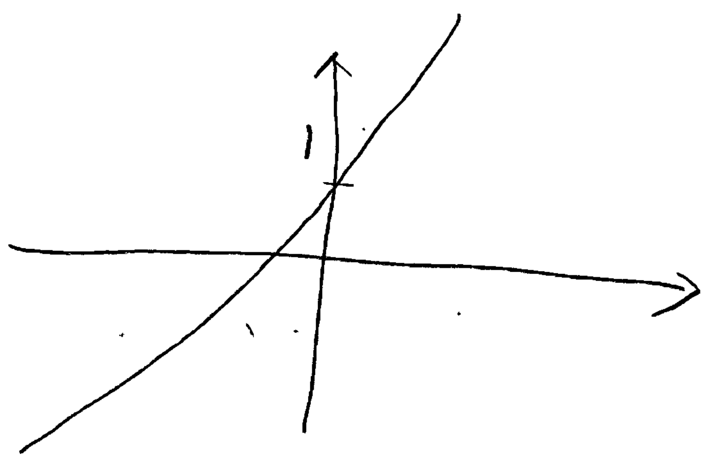
$y' = \frac{2}{x}$,

tangent line : $y = \frac{2}{e}x + 1$

3. Sketch the graph of $f(x) = x + e^x$ using the Curve Sketching techniques you learned in Chapter 3.

$f'(x) = 1 + e^x$. No critical number.

$f''(x) = e^x$. No inflection point.



4. Find y' if $2e^y + \ln(xy) = 2x^2y + 4$.

$\ln(xy) = \ln x + \ln y$.

$$2e^y \frac{dy}{dx} + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 4xy + 2x^2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(2e^y + \frac{1}{y} - 2x^2\right) = 4xy - \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{4xy - \frac{1}{x}}{2e^y + \frac{1}{y} - 2x^2}.$$

5. Find a formula for the n-th derivative of $g(s) = e^{4s}$.

$g'(s) = 4e^{4s}$. $g''(s) = 4^2 e^{4s}$.

$g^{(n)}(s) = 4^n e^{4s}$.

6. Compute the following integrals.

(a) $\int_0^{\frac{e-1}{2}} \frac{5}{1+2x} dx$

$= \frac{5}{2} \ln(1+2x) \Big|_0^{\frac{e-1}{2}} = \frac{5}{2} \ln(1 + e - 1)$

$= \frac{5}{2} \ln e = \frac{5}{2}$

(b) $\int \frac{\sin(\ln x)}{x} dx$

$u = \ln x$

$\int \sin u \, du = -\cos u + C = -\cos(\ln x) + C$

(c) $\int_1^e \frac{(\ln t)^4}{t} dt$

$\ln t = u$ $\int_0^1 u^4 du = \frac{1}{5}$

(d) $\int_0^{\ln(1+\pi)} e^x \cos(1-e^x) dx$

$u = 1 - e^x$

$du = -e^x dx$

$\int_0^{-\pi} -\cos u \, du = -\sin u \Big|_0^{-\pi} = 0.$

(e) $\int \frac{\log_{10} x}{x} dx$ $= \int \frac{1}{\ln 10} \cdot \frac{\ln x}{x} dx$

$U = \ln x \Rightarrow \quad = \frac{1}{\ln 10} \int u\,du = \frac{1}{\ln 10} \frac{1}{2} u^2 + C$

$$= \frac{(\ln x)^2}{2\ln 10} + C$$

7. Solve the inequality $1 < e^{4x-2} < 2$, for x.

Take ln.

$0 = \ln 1 < 4x - 2 < \ln 2$

$\Rightarrow \quad 2 < 4x < \ln 2 + 2$

$\Rightarrow \quad \frac{1}{2} < x < \frac{\ln 2 + 2}{4}$

8. Solve the following equations:

(a) $e^{4x-6} = 8$.

Take ln. $4x - 6 = \ln 8$

$x = \frac{\ln 8 + 6}{4}$

(b) $e - e^{-4x} = 4$.

$e^{-4x} = e - 4$. Since $e < 3$, $e - 4 < 0$.

Thus, the solution does not exist.

(c) $\ln(x) + \ln(x - 1) = 1$.

$\ln x + \ln(x-1) = \ln(x(x-1)) = 1$

$\Rightarrow e^{\ln(x(x-1))} = x(x-1) = e$.

Use the quadratic formula!

9. Differentiate the following functions:

(a) $G(x) = 4^{C/x}$, where C is a constant

By Chain rule,

$G'(x) = \left(e^{\frac{C\ln 4}{x}}\right)' = e^{\frac{C\ln 4}{x}} \cdot \left(-\frac{C\ln 4}{x^2}\right)$

(b) $y = x^x = e^{x\ln x}$

$y' = e^{x\ln x} \cdot (x\ln x)' = e^{x\ln x}(\ln x + 1)$

(c) $y = (\sin x)^{\ln x} = e^{\ln x \cdot \ln\sin x}$

$y' = e^{\ln x \cdot \ln\sin x} \cdot (\ln x \cdot \ln\sin x)'$

$= e^{\ln x \cdot \ln\sin x} \cdot \left(\frac{1}{x}\ln\sin x + \frac{\cos x}{\sin x}\ln x\right)$

(d) $y = (3x^2+5)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(3x^2+5)}$

$y' = e^{\frac{1}{x}\ln(3x^2+5)} \cdot \left(\frac{1}{x}\ln(3x^2+5)\right)'$

$= e^{\frac{1}{x}\ln(3x^2+5)} \left(-\frac{1}{x^2}\ln(3x^2+5) + \frac{6x}{x(3x^2+5)}\right)$

10. Find y' if $x^y = y^x$.

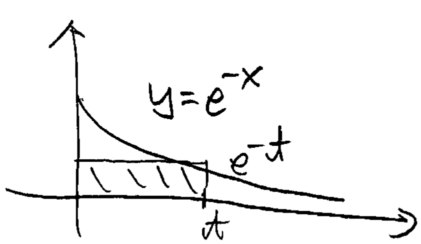
$e^{y\log x} = e^{x\log y}$

$\Rightarrow e^{y\log x}\frac{d}{dx}(y\log x) = e^{x\log y}\frac{d}{dx}(x\log y)$

$\Rightarrow x^y\left(y'\log x + \frac{y}{x}\right) = y^x\left(\log y + \frac{x}{y}y'\right)$

Solve this for y'!

11. A computer is programmed to inscribe a series of rectangles in the first quadrant under the curve of $y = e^{-x}$. What is the area of the largest rectangle that can be inscribed?



Want to maximize $f(t) = te^{-t}$.

$f'(t) = e^{-t} - te^{-t} = (1-t)e^{-t} = 0$

$\Rightarrow t = 1$.

Since $f(0) = 0$, $\lim_{t\to\infty} f(t) = 0$,

$f(1) = e^{-1}$ is the largest area.

$a^x = e^{x\ln a}$

12. Let $a \neq -1$ be a constant. Calculate $\int \frac{x}{a} + \frac{a}{x} + x^a + a^x + ax\, dx$.

$$\frac{1}{a}\cdot\frac{1}{2}x^2 + a\ln x + \frac{x^{a+1}}{a+1} + \frac{1}{\ln a}e^{x\ln a} + \frac{a}{2}x^2 + C$$

13. Sketch the graph of $f(x) = \ln(1+x^2)$ using the Curve Sketching techniques you learned in Chapter 3.

$$f'(x) = \frac{2x}{1+x^2} = 0 \Rightarrow x = 0.$$

$$f''(x) = \frac{2(1+x^2)-4x^2}{(1+x^2)^2} = \frac{2(1-2x^2)}{(1+x^2)^2} = 0$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

		$-\sqrt{2}$		0		$\sqrt{2}$	
f	$\searrow$		$\searrow$		$\nearrow$		$\nearrow$
f'	$-$		$-$	0	$+$		$+$
f''	$-$	0	$+$		$+$	0	$-$