KEY

Math 221 - Week 12 - Worksheet 1
Topics: Section 6.6 - Inverse Trigonometric Functions

Instructions: Listen to your TA's instructions. There are substantially more problems on this worksheet than we expe to be done in discussion, and your TA might not have you do problems in order. The worksheets are intentionally long than will be covered in discussion in order to give students additional practice problems they may use to study. Do n worry if you do not finish the worksheet :).

1. Determine the derivatives of the following functions:

(a) $f(x) = \sin^{-1}(4x^2)$

Use $\frac{d}{dx}\left[\arcsin(u)\right] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ to obtain

$$f'(x) = \frac{1}{\sqrt{1-(4x^2)^2}}(8x) = \frac{8x}{\sqrt{1-16x^4}}$$

(b) $g(s) = \cos^{-1}(s)\ln(2s)$.

Use $\frac{d}{dx}\left[\arccos(x)\right] = -\frac{1}{\sqrt{1-x^2}}$ to obtain

$$g'(s) = \left(-\frac{1}{\sqrt{1-s^2}}\right)(\ln(2s)) + (\arccos(s))\left(\frac{1}{2s}\cdot 2\right) = \frac{1}{s}\arccos(s) - \frac{\ln(2s)}{\sqrt{1-s^2}}$$

(c) $y = (\tan^{-1}x)^2$

Use $\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}$ to obtain

$$\frac{dy}{dx} = 2(\arctan(x))\left(\frac{1}{1+x^2}\right)$$

(d) $f(x) = \arcsin(e^x)$

$$f'(x) = \frac{1}{\sqrt{1-(e^x)^2}}\cdot(e^x) = \frac{e^x}{\sqrt{1-e^{2x}}}$$

(e) $y = \arctan\sqrt{\frac{1-x}{1+x}}$

Use $\frac{d}{dx}\left[\arctan(u)\right] = \frac{1}{1+u^2}\frac{du}{dx}$ to obtain

$$\frac{dy}{dx} = \frac{1}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\left(\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left(\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right)\right) = \left(\frac{1}{1+\frac{1-x}{1+x}}\right)\left(-\sqrt{\frac{1+x}{1-x}}\left(\frac{1}{(1+x)^2}\right)\right)$$

$$= -\frac{(1+x)^{3/2}}{2\sqrt{1-x}}\left(\frac{1}{(1+x)^2}\right) = -\frac{1}{2\sqrt{(1-x)(1+x)}} = -\frac{1}{2\sqrt{1-x^2}}$$

(f) $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

$$\frac{dy}{dx} = \frac{1}{1+\frac{x^2}{a^2}} + \frac{1}{\sqrt{\frac{x-a}{x+a}}}\left(\frac{1}{2}\left(\frac{x-a}{x+a}\right)^{-\frac{1}{2}}\left(\frac{(x+a)(1)-(x-a)(1)}{(x+a)^2}\right)\right)$$

$$= \frac{1}{1+\frac{x^2}{a^2}} + \frac{1}{2}\frac{1}{\left(\frac{x-a}{x+a}\right)}\left(\frac{2a}{(x+a)^2}\right) = \frac{1}{1+\frac{x^2}{a^2}} + \frac{a}{(x-a)(x+a)}$$

2. Find the absolute max and absoulte min of the function $f(x) = e^x - ex$ on the interval $0 \le x \le 5$.

$f'(x) = e^x - e$

$f'(x) = 0$ at $x = 1$.

Sign of $f'(x)$

x	0		1		5
		$-$		$+$	

So $f'(1)$ is a local min.

$f(0) = 1$ $\quad f(5) = e^5 - 5e > 1$

$f(1) = 0$

So,

Abs max of $f(x)$ is $e^5 - 5e$ at $x = 5$.
Abs min of $f(x)$ is 0 at $x = 1$.

3. Find y' if $\tan^{-1}(x^2y) = 2x + xy$.

Use $\frac{d}{dx}\left[\tan^{-1}(u)\right] = \frac{1}{1+u^2}\frac{du}{dx}$ to obtain

$$\frac{1}{1+x^4y^2}\left(2xy + x^2\frac{dy}{dx}\right) = 2 + \left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{2xy}{1+x^4y^2} - 2 - y\right)\left(\frac{1}{x - \frac{x^2}{1+x^4y^2}}\right) = \frac{dy}{dx}$$

4. Find and equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

Use $\frac{d}{dx}\left[\cos^{-1}(u)\right] = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$ to obtain

$$\frac{dy}{dx} = -\frac{3}{2}\left(\frac{1}{\sqrt{1 - x^2/4}}\right)$$

Thus, $m = -\frac{3}{2}\left(\frac{1}{\sqrt{1-1/4}}\right) = -\frac{1}{\sqrt{3}}$. So the tangent line is given by $y - \pi = -\frac{1}{\sqrt{3}}(x-1)$

5. Show that there is exactly one root of the equation $\ln(x) = 3 - x$ and that it lies between 1 and e.

We will use IVT on $f(x) = \ln(x) + x - 3$ to show $f(x)$ has a root b/w 1 and e (which we can do since $f(x)$ is cont. for $x > 0$).

$f(1) = 0 + 1 - 3 = -2$

$f(e) = 1 + e - 3 > 0$

Thus, by IVT there is a c b/w 1 and e where $f(c) = \ln(c) + c - 3 = 0$. Thus, $\ln(c) = 3 - c$ as desired.

6. Evaluate the following integrals.

(a) $\int \frac{1}{(y-1)^2+1}dy$

Let $u = y - 1$, so $du = dy$. The integral becomes

$$\int \frac{1}{u^2+1}\,du = \arctan(u) + C = \arctan(y-1) + C$$

(b) $\displaystyle\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2}$

Let $u = 4x$, so $du = 4dx$.

$$\Rightarrow \int_0^{\sqrt{3}} \frac{\frac{1}{4}du}{1+u^2} = \frac{1}{4}\left(\tan^{-1}(u)\right)\Big|_{u=0}^{\sqrt{3}} = \frac{1}{4}\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(0)\right)$$
$$= \frac{1}{4}\tan^{-1}(\sqrt{3}) = \frac{\pi}{12}$$

(c) $\displaystyle\int \frac{1+x}{1+x^2}dx$

$$\int \frac{1+x}{1+x^2}dx = \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx = \arctan(x) + C + \int \frac{x}{1+x^2}dx$$

Let $u = 1+x^2$ so $\frac{1}{2}du = x\,dx$. Then

$$= \arctan(x) + C + \frac{1}{2}\int \frac{1}{u}du = \arctan(x) + \frac{1}{2}\ln|u| + C = \arctan + \frac{1}{2}\ln|1+x^2| + C$$

(d) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

Let $u = \cos(x)$, so $du = -\sin(x)\,dx$

$$\Rightarrow -\int_1^0 \frac{1}{1+u^2}du = -\left(\tan^{-1}(u)\right)\Big|_{u=1}^{0} = -\left(\tan^{-1}(0) - \tan^{-1}(1)\right)$$
$$= -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}$$

(e) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}\sin^{-1}x} =$

Let $u = \sin^{-1}(x)$ so $du = \frac{1}{\sqrt{1-x^2}}dx$

$$\Rightarrow \int \frac{1}{u}du = \ln|u| + C = \ln|\sin^{-1}(x)| + C$$

(f) $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}dx$

$$\Rightarrow 8\left(\tan^{-1}(x)\right)\Big|_{x=1/\sqrt{3}}^{\sqrt{3}} = 8\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}(1/\sqrt{3})\right) = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{4\pi}{3}$$

(g) $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

Let $u = e^{2x}$, so $du = 2e^{2x}dx$

$$\Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du = \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(e^{2x}) + C$$

(h) $\int \frac{x}{1+x^4}dx$

Let $u = x^2$, so $du = 2x dx$

$$\Rightarrow \frac{1}{2}\int \frac{1}{1+u^2}du = \frac{1}{2}\arctan(u) + C = \frac{1}{2}\arctan(x^2) + C$$

(i) $\int \frac{1}{\sqrt{a^2-x^2}}dx$ for $a > 0$

$$\Rightarrow \int \frac{1}{a\sqrt{1-\frac{x^2}{a^2}}}dx$$

Let $u = \frac{x}{a}$, so $du = \frac{1}{a}dx$

$$\Rightarrow \int \frac{1}{\sqrt{1-u^2}}du = \arcsin(u) + C = \arcsin\left(\frac{x}{a}\right) + C$$

(j) $\int \frac{\sin(\arctan(x))}{2+2x^2}dx$

$$\Rightarrow \int \frac{\sin(\tan^{-1}(x))}{2(1+x^2)}dx$$

Let $u = \tan^{-1}(x)$, so $du = \frac{1}{1+x^2}dx$.

$$\Rightarrow \frac{1}{2}\int \sin(u)\,du = \frac{1}{2}\cos(u) + C = \frac{1}{2}\cos(\tan^{-1}(x)) + C$$

7. Find $\frac{dq}{dp}$ if $\arcsin(pq) + q^2 = \frac{q}{p}$.

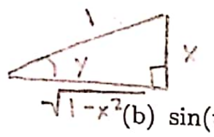
$$\Rightarrow \frac{1}{\sqrt{1-p^2q^2}}\left(p\frac{dq}{dp} + q\right) + 2q\frac{dq}{dp} = \frac{p\frac{dq}{dp} - q}{p^2}$$

$$\Rightarrow \left(\frac{p^2q}{\sqrt{1-p^2q^2}} + q\right)\left(\frac{1}{p - 2q - \frac{p}{\sqrt{1-p^2q^2}}}\right) = \frac{dq}{dp}$$

8. Eliminate the trig functions from the following expressions:

(a) $\tan(\sin^{-1} x)$

Let $\sin^{-1}(x) = y$. We have the diagram (since $\sin(y) = x$)

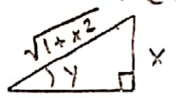


$, \ -\frac{\pi}{2} < y < \frac{\pi}{2}$

So $\tan(\sin^{-1}(x)) = \tan(y) = \frac{x}{\sqrt{1-x^2}}$

(b) $\sin(\tan^{-1} x)$

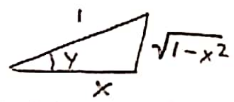
Let $\tan^{-1}(x) = y$, so $x = \tan(y)$. We have the diagram



So $\sin(\tan^{-1}(x)) = \sin(y) = \frac{x}{\sqrt{1+x^2}}$

(c) $\sin(2 \arccos x)$

Let $\cos^{-1}(x) = y$, so $x = \cos(y)$. We have the diagram



Using $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we get $\sin(2\arccos(x)) = \sin(2y) = 2\sin(y)\cos(y)$
$= 2x\sqrt{1-x^2}$

9. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16 - x^2}$, find the equation of the line tangent to $g(x)$ at $x = 2$.

$$g'(x) = \left(x\left(\frac{1}{\sqrt{1-x^2/16}} \cdot \frac{1}{4}\right) + \sin^{-1}(x/4)\right) + \frac{1}{2}(16-x^2)^{-1/2}(-2x)$$

$$\text{So } g'(2) = \left(2\left(\frac{2}{\sqrt{3}} \cdot \frac{1}{4}\right) + \frac{\pi}{3}\right) + \frac{1}{2}\left(\frac{1}{2\sqrt{3}}\right)(-4)$$

$$= \frac{1}{\sqrt{3}} + \frac{\pi}{3} - \frac{1}{\sqrt{3}} = \frac{\pi}{3}$$

So the tangent line is given by the equation

$$y - g(2) = \frac{\pi}{3}(x-2) \Rightarrow y - \left(\frac{2\pi}{3} + 2\sqrt{3}\right) = \frac{\pi}{3}(x-2)$$

10. Sketch the function $f(x) = \tan^{-1}(x) - x$ using the techniques you learned in Chapter 3.

$f(0) = \tan^{-1}(0) - 0 = 0$

Y-Int: $(0,0)$

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

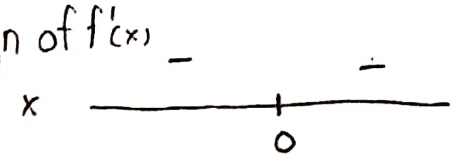
$f(x) = \tan^{-1}(x) + x = 0$ where $\tan(x) = x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$x = 0$ satisfies this (see above). There may be others, but let's see what we can do without them.

$f'(x) = \frac{1}{1+x^2} - 1$

$f'(x) = 0$ at $x = 0$

Sign of $f'(x)$



$f''(x) = \frac{-2x}{(1+x^2)^2}$

$f''(x) = 0$ at $x = 0$

Sign of $f''(x)$