Math 221 Worksheet 1
December 1, 2020
Section 6.8: L'Hôpital's Rule; Section 5.1: Areas Between Curves

1. For each of the following, evaluate the limit or show that it does not exist.

(a) $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

Plugging in $x = 0$ gives the indeterminate form $\frac{0}{0}$, so by L'Hôpital's rule,

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x}.$$

This again gives $\frac{0}{0}$, so by L'Hopital, $\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$

(b) $\lim_{x\to 1} \frac{1-x}{1+\cos(x)}$

The function $\frac{1-x}{1+\cos(x)}$ is continuous at $x = 1$ (since $1 + \cos(1) \neq 0$), so

$$\lim_{x\to 1} \frac{1-x}{1+\cos(x)} = \frac{1-1}{1+\cos(1)} = \boxed{0}$$

(c) $\lim_{x\to 0^+} \sqrt{x}\ln x$

"Plugging in" $x = 0$ gives the indeterminant form $0\cdot(-\infty)$. Convert the product to a quotient and apply L'Hôpital:

$$\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} \left(-2x^{1/2}\right) = \boxed{-\infty}$$

(d) $\lim_{x\to\infty} (\ln x)^{\frac{1}{x}}$

First, rewrite in base e (good to do whenever you see an exponential function):

$$(\ln x)^{\frac{1}{x}} = e^{\frac{\ln(\ln x)}{x}}$$

So $\lim_{x\to\infty} (\ln x)^{\frac{1}{x}} = e^{\lim_{x\to\infty} \frac{\ln(\ln x)}{x}}$ provided the second limit exists.

By L'Hôpital,

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1} = 0.$$

So the answer is $e^0 = \boxed{1}$

(e) $\lim_{t \to 0} \frac{e^{3t} - 1}{\sin(t)}$

By L'Hôpital,

$$\lim_{t \to 0} \frac{e^{3t} - 1}{\sin(t)} = \lim_{t \to 0} \frac{e^{3t} \cdot 3}{\cos(t)} = \frac{e^{3 \cdot 0} \cdot 3}{\cos(0)} = \boxed{3}$$

(f) $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}}$

By L'Hôpital,

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2} x^{-1/2}} = \lim_{x \to \infty} 2x^{-1/2} = \boxed{0}$$

(g) $\lim_{x \to 1^+} \left[\ln(x^7 - 1) - \ln(x^5 - 1)\right]$

First make the difference a quotient:

$$\ln(x^7 - 1) - \ln(x^5 - 1) = \ln\left(\frac{x^7 - 1}{x^5 - 1}\right)$$

Then $\lim_{x \to 1^+} \left(\ln(x^7 - 1) - \ln(x^5 - 1)\right) = \ln\left(\lim_{x \to 1^+} \frac{x^7 - 1}{x^5 - 1}\right)$ provided the second limit exists. By L'Hôpital,

$$\lim_{x \to 1^+} \frac{x^7 - 1}{x^5 - 1} = \lim_{x \to 1^+} \frac{7x^6}{5x^4} = \lim_{x \to 1^+} \frac{7}{5} x^2 = \frac{7}{5}.$$

So the answer is $\boxed{\ln\left(\frac{7}{5}\right)}$

(h) $\lim_{y \to 0} \frac{\sin y}{y + \tan y}$

By L'Hôpital,

$$\lim_{x \to 0} \frac{\sin(y)}{y + \tan(y)} = \lim_{y \to 0} \frac{\cos(y)}{1 + \sec^2(y)} = \frac{\cos(0)}{1 + \sec^2(0)} = \boxed{\frac{1}{2}}$$

(i) $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x$

Like in part (d): $\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x = \lim_{x \to \infty} e^{x \ln\left(1 + \frac{4}{x}\right)} = e^{\lim_{x \to \infty} x \ln\left(1 + \frac{4}{x}\right)}$

By L'Hôpital,

$$\lim_{x \to \infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x \to \infty} \frac{\ln\left(1 + \frac{4}{x}\right)}{x^{-1}} = \lim_{x \to \infty} \frac{\frac{1}{1 + \frac{4}{x}} \cdot \frac{-4}{x^2}}{-x^{-2}} = \lim_{x \to \infty} \frac{4}{1 + \frac{4}{x}} = 4.$$

So the answer is $\boxed{e^4}$

2. Determine values of a and b such that $\lim_{x\to 0}\left(\frac{\sin(2x)}{x^3} + 2b + \frac{a}{x^2}\right) = 0$.

$$\lim_{x\to 0}\left(\frac{\sin(2x)}{x^3} + 2b + \frac{a}{x^2}\right) = 2b + \lim_{x\to 0}\frac{\sin(2x) + ax}{x^3}$$ provided the second limit exists. By L'Hôpital,

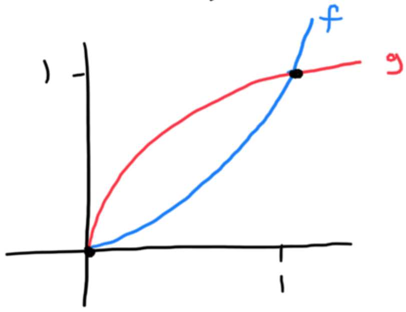
$$\lim_{x\to 0}\frac{\sin(2x)+ax}{x^3} = \lim_{x\to 0}\frac{2\cos(2x)+a}{3x^2} \quad ☆$$

so we must have $\boxed{a=-2}$ (otherwise the limit does not exist). Applying L'Hôpital twice more to ☆, we find that ☆ $= -\frac{4}{3}$. Since the total limit is 0, we must have $\boxed{b = \frac{2}{3}}$

3. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$

(a) Find all points where the graphs of f and g intersect. Sketch the graphs.

$x^2 = \sqrt{x} \iff x = 0$ or $x = 1$, so the intersections are $(0,0)$ and $(1,1)$.



(b) Find the area of the bounded region(s) enclosed by the graphs of f and g.

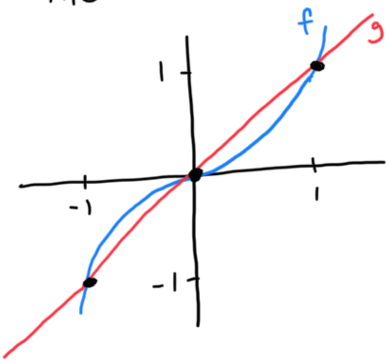
$$\text{area} = \int_0^1 g(x)\,dx - \int_0^1 f(x)\,dx$$
$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$
$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

4. Repeat Problem 3 for the following pairs of functions:

(a) $f(x) = x^3$ and $g(x) = x$

$x^3 = x \iff x = -1,\ x = 0,$ or $x = 1$.

The intersections are $(-1,-1)$, $(0,0)$, and $(1,1)$.

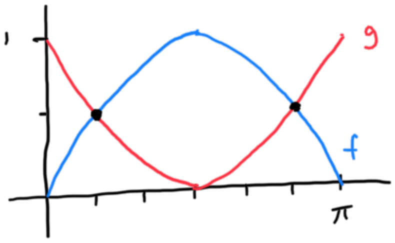


$$\text{area} = \int_{-1}^0 (f(x) - g(x))\,dx + \int_0^1 (g(x) - f(x))\,dx$$
$$= \int_{-1}^0 (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$
$$= -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{4}$$
$$= \boxed{\frac{1}{2}}$$

(b) $f(x) = \sin(x)$ and $g(x) = 1 - \sin(x)$ for $0 \leq x \leq \pi$

$\sin(x) = 1 - \sin(x) \iff x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

Intersections are $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{1}{2}\right)$.

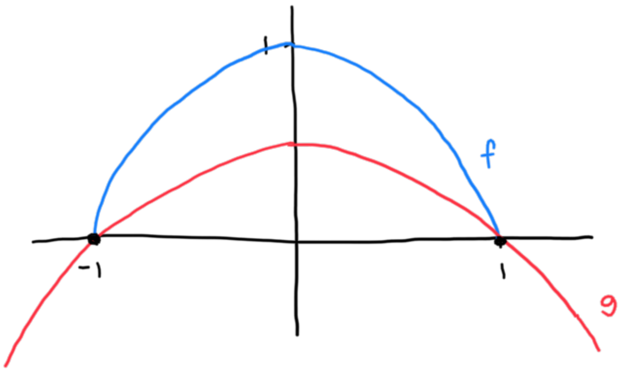


$$\begin{aligned}
\text{area} &= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (f(x) - g(x))\, dx \\
&= \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (2\sin(x) - 1)\, dx \\
&= -2\cos\left(\frac{5\pi}{6}\right) - \frac{5\pi}{6} + 2\cos\left(\frac{\pi}{6}\right) + \frac{\pi}{6} \\
&= \boxed{2\sqrt{3} - \frac{2\pi}{3}}
\end{aligned}$$

(c) $f(x) = \sqrt{1-x^2}$ and $g(x) = \dfrac{1-x^2}{2}$ for $-1 \leq x \leq 1$

$\sqrt{1-x^2} = \frac{1-x^2}{2} \iff x = -1$ or $x = 1$.

Intersections are $(-1, 0)$ and $(1, 0)$.



The graph of f is a semicircle of radius 1. So

$$\begin{aligned}
\text{area} &= \frac{\pi}{2} - \int_{-1}^{1} g(x)\, dx \\
&= \frac{\pi}{2} - \int_{-1}^{1} \frac{1-x^2}{2}\, dx \\
&= \frac{\pi}{2} - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{2} - \frac{1}{6}\right) \\
&= \boxed{\frac{\pi}{2} - \frac{2}{3}}
\end{aligned}$$