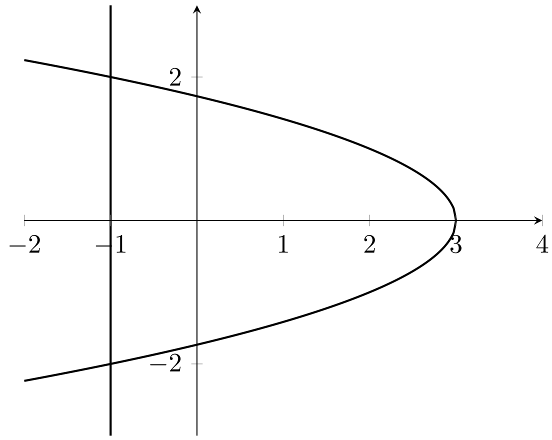
221 - Week 13 Worksheet 2 solutions

1. (a) 

In order to find the intersection points we can simply put $x = -1$ into the first equation: $-1 = 3 - y^2$, which we can rearrange to get $y^2 = 4$ so our two intersection points are $(-1, 2)$ and $(-1, -2)$.

(b) First lets try to set up the integral we should get while integrating with respect to y, in order to do this we need to find the lower and upper bounds for y as functions of x. Looking at our graph we see that the lower bound will be the vertical line $x = -1$ and the upper bound will be the parabola $x = 3 - y^2$. Moreover the lowest value of y is at the lower intersection point and is therefore -2 while the highest value is 2 from the other intersection point. Hence the integral for our are is

$$\int_{-2}^{2} 3 - y^2 - (-1)\, dy = \int_{-2}^{2} 4 - y^2\, dy$$

For the second way we want to integrate with respect to x so we need to find the lower and upper bounding curves as functions of x. This is slightly harder to see but from the graph the lower bound is the 'lower half' of the curve $x = 3 - y^2$ whereas the upper curve is the 'upper half' of this curve. If we solve this curve for y in terms of x we get $y = \pm\sqrt{3-x}$ so the lower curve is $y = -\sqrt{3-x}$ and the upper curve is $y = \sqrt{3-x}$. From the graph we can see easily that the lowest value for x is -1 and the largest is 3. Thus the second integral for our area is

$$\int_{-1}^{3} \sqrt{3-x} - (-\sqrt{3-x})\, dx = \int_{-1}^{3} 2\sqrt{3-x}\, dx$$

(c) The first integral is straightforward:

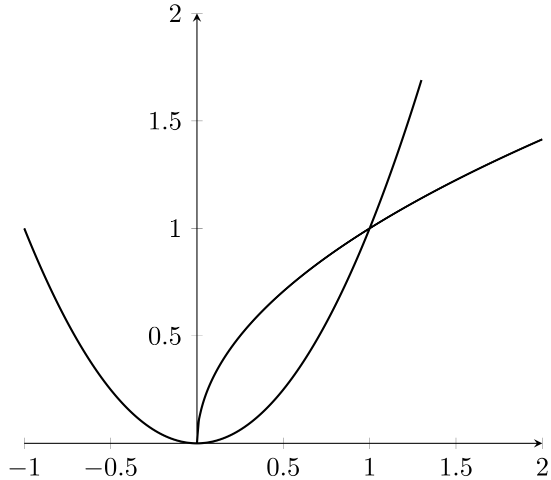
$$\int_{-2}^{2} 4 - y^2\, dy = [4y - y^3/3]_{-2}^{2} = (4(2) - 2^3/3) - (4(-2) - (-2)^3/3) = 8 - 8/3 + 8 - 8/3 = 16 - 16/3 = 32/3$$

The second integral looks more complicated so we should probably try a u-substitution, the natural choice is $u = 3 - x$, then $du = -dx$. Since we are working with a definite integral we need to be careful of bounds, I will change the x bounds into u bounds and use my new bounds for the rest of the problem: when $x = -1$, $u = 3 - (-1) = 4$ and when $x = 3$, $u = 3 - 3 = 0$ So when we complete the substitution we get

$$\int_{4}^{0} 2\sqrt{u}(-1)\, du = [-4/3u^{3/2}]_4^0 = -4/3(0) - (-4/3(4^{3/2})) = 32/3$$

Which is the same answer we got before!

2. (a) To find the intersection points we set $\sqrt{x} = x^2$, squaring this equation yields $x = x^4$ which we can easily solve to get $x = 0, 1$. We can then sketch the graph:



(b) The upper curve here is $y = \sqrt{x}$ and the lower curve is $y = x^2$ so our integral is

$$\int_0^1 \sqrt{x} - x^2 \, dx$$

(c)

$$y = \sqrt{x} \iff y^2 = x$$

So one of our equations is $x = y^2$. For the other one we need to take a square root, since we are in the first quadrent we should take the positive one so we get $x = \sqrt{y}$.

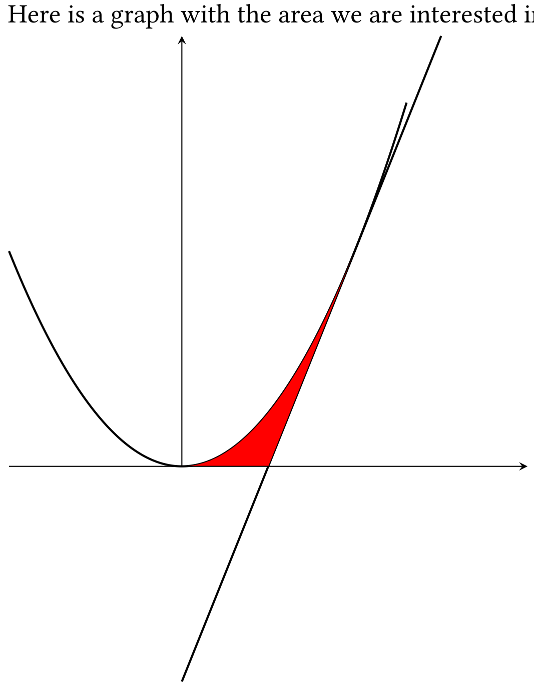
(d)

$$\int_0^1 \sqrt{y} - y^2 \, dy$$

3. (a) The derivative of x^2 is $2x$ so the tangent line at the point (a, a^2) will be given by

$$y - a^2 = 2a(x - a)$$

(b) Here is a graph with the area we are interested in shaded in red:

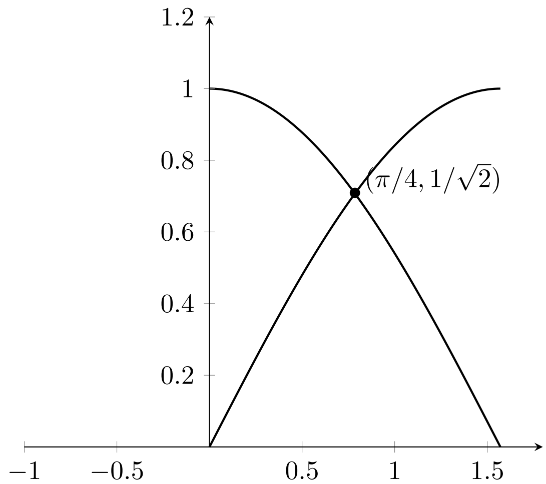


(c) Looking at the graph it seems that it will be easier to integrate with respect to y since in that case we won't have to break up the area into two seperate parts. The bounds for y will be 0 and a^2. For the curves we need to solve both of the equations for x in terms of y. The first we have already seen $x = \sqrt{y}$ and the second is linear so we can solve it directly to get $x = \frac{1}{2a}(y + a^2)$ so the integral for our area will be

$$\int_0^{a^2} \frac{1}{2a}(y+a^2) - \sqrt{y}\, dy = [\frac{1}{2a}(y^2/2 + ya^2) - 2/3y^{3/2}]_0^{a^2} = \frac{1}{2a}(a^4/2 + a^4) - 2/3a^3 = a^3/12$$

And we want this area to be $2/3$ so we set $a^3/12 = 2/3$ which we can solve to get $a = 2$.

4. (a) Equating the functions gives $\sin x = \cos x$ and since this clearly can't happen when $\cos x = 0$ we can divide by it to get $\tan x = 1$ whose only solution on the given interval is $x = \pi/4$.

(b) 

(c) Looking at our picture above we see that we are going to have to split our integral into two parts since the two curves switch places. The first part is from $x = 0$ to $x = \pi/4$ and here the curve $y = \cos x$ is on the top whereas $y = \sin x$, so this integral will be

$$\int_0^{\pi/4} \cos x - \sin x\, dx$$

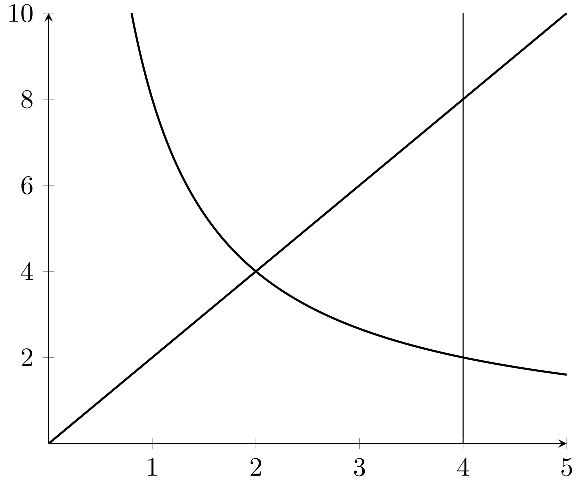
For the second part we will be going from $x = \pi/4$ to $x = \pi/2$ and this time the curve $y = \sin x$ is on top and $y = \cos x$ is on the bottom so the integral here will be

$$\int_{\pi/4}^{\pi/2} \sin x - \cos x\, dx$$

So our final answer will be the sum of these two integrals:

$$\int_0^{\pi/4} \cos x - \sin x\, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x\, dx$$

5. (a)



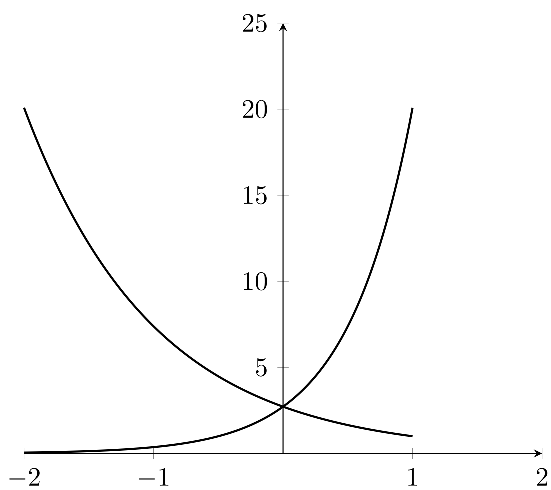
So the area will be

$$\int_2^4 2x - 8/x\, dx = [x^2 + 8/x^2]_2^4 = 16 + 8/16 - (4 + 8/4) = 10.5$$

(b) The curve $x = 3 + y^2$ will be the upper curve so the area will be given by

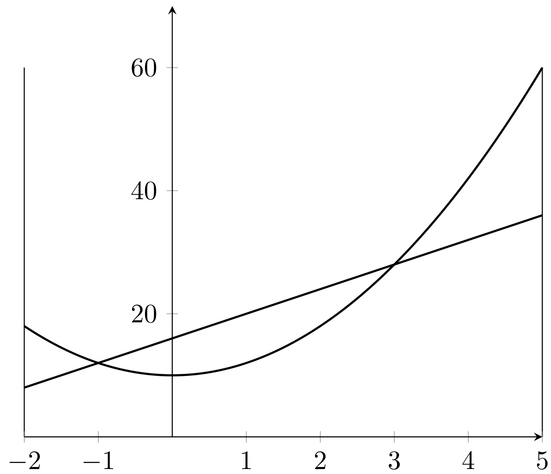
$$\int_{-2}^{1} 3 + y^2 - (2 - y^2)\, dy = \int_{-2}^{1} 1 + 2y^2 = [y + 2y^3/3]_{-2}^{1} = 1 + 2/3 + 2 + 16/3 = 9$$

(c) Here a picture will be helpful:



From this we see that we need to deal with the cases where $x > 0$ and $x < 0$ seperately since the curves cross at that point. So our area will be given by

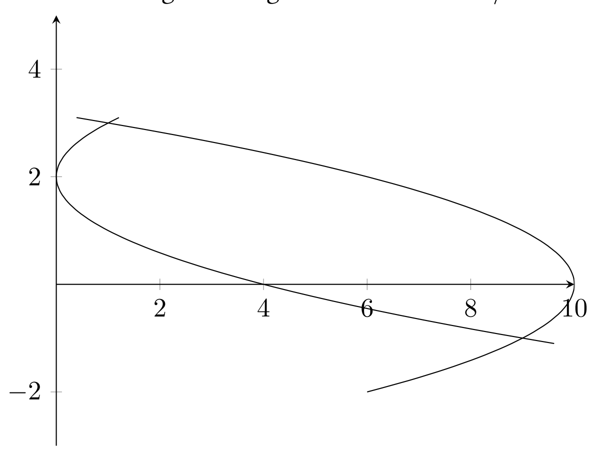
$$\int_{-2}^{0} e^{1-x} - e^{1+2x}\, dx + \int_{0}^{1} e^{1+2x} - e^{1-x}\, dx = [-e^{1-x} - \frac{1}{2}e^{1+2x}]_{-2}^{0} + [\frac{1}{2}e^{1+2x} + e^{1-x}]_0^1 = \frac{1}{2}e^{-5} + \frac{3}{2}e^3 + 1 - 3e$$

(d) 

Here we need three parts, first we compute that the two intersection points are given by $x = -1$ and $x = 3$. So

$$\int_{-2}^{-1} 2x^2 + 10 - (4x + 16)\, dx + \int_{-1}^{3} 4x + 16 - (2x^2 + 10)\, dx + \int_{3}^{5} 2x^2 + 10 - (4x + 16)\, dx$$

After evaluating this we get an answer of $142/3$

(e) 

We can solve for the intersection points to get $y = 3$ and $y = -1$ so these will be the bounds for the integral with respect to y. So we get

$$\int_{-1}^{3} -y^2 + 10 - (y-2)^2\, dy = \int_{-1}^{3} -2y^2 + 4y + 6\, dy = [-2y^3/3 + 2y^2 + 6y]_{-1}^{3} = 64/3$$

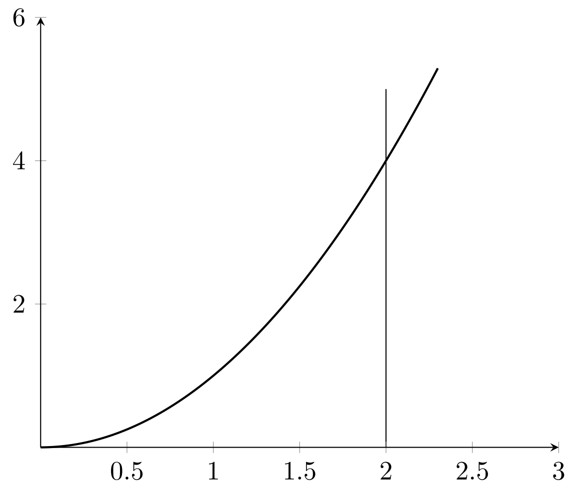
(f) This one may be tricky since $y = xe^{-x^2}$ is probably not something you immediately know how to graph. We could try to algeraically find the intersection points since that is the most important thing but this is also difficult. In liu of trying directly note that if we plug in 0 to both we get $y = 0e^0 = 0$ and $y = 0 + 1 = 1$ and if we plug in $x = 2$ then we get $y = 2e^{-4}$ and $y = 3$. So we can see that in both cases the first curve($y = xe^{-x^2}$) lies below the second one. So we might hope that the second curve is the upper one across the whole region we are interested in. Indeed this will be the case, in order to see this we can compute the derivative of both curves $y' = e^{-x^2} - 2x^2e^{-x^2}$ and $y' = 1$ and note that the derivative of the second function is always greater or equal to the derivative of the first on the interval $[0, 2]$. Hence the curve $y = x + 1$ must always be on top.

Now that we know this the integral is relatively easy to set up:

$$\int_{0}^{2} x + 1 - (xe^{-x^2})\, dx$$

Now all that's left is to compute this which we can do using a substitution to deal with the exponential in order to find that the area is equal to

$$[x^2/2 + x + \frac{1}{2}e^{-x^2}]_0^2 = 4 + \frac{e^{-4} - 1}{2}$$



6. (a) Looking at the graph of the base above we see that the radius of the vertical semicercles will be given simply by $r = y = x^2$. So recalling that the area of a semicircle is given by $\pi r^2/2$ we can set up an integral for the volume:

$$\int_0^2 \frac{\pi}{2}(x^2)^2\,dx = [\frac{\pi x^5}{10}]_0^2 = \frac{16\pi}{5}$$

   (b) Here we will repeat the above only now since we are looking horizontally the radius will be given by the difference of two curves $r = 2 - \sqrt{y}$. So the integral will be

$$\int_0^4 \frac{pi}{2}(2 - \sqrt{y})^2\,dy = \int_0^4 \frac{\pi}{2}(4 - 4\sqrt{y} + y)\,dy = 4\pi/3$$

7. The equation of the boundary of the disk is $x^2 + y^2 = r^2$. If we chose to work with vertical cross sections then the length of the base of the triangles will be given by the upper half semicircle minus the lower one: $s = \sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2}) = 2\sqrt{r^2 - x^2}$. So to find the volume of the shape we need to integrate the formula for area given in the question from $x = -r$ to $x = r$.

$$\int_{-r}^{r} \frac{\sqrt{3}}{4}(2\sqrt{r^2 - x^2})^2\,dx = \int_{-r}^{r} \sqrt{3}(r^2 - x^2)\,dx = \frac{4\sqrt{3}}{3}$$

8. The easiest way to approach this problem is to take cross sections parallel to the base, so in particular we will be integrating with respect to whichever variable we want to use to denote height, I will use z. The advantage of this approach is that all of our cross sections will be equilateral triangles and so it will be much easier to write a formula for the area of the cross sections which we need in order to integrate for the volume. Moreover we know that following this the bounds in our integral will be 0 and h.

   Since we already have a formula for the area of an equilateral triangle in terms of it's side length we just need to figure out how this side length varies with z. The easiest approach here is to use similar triangles. If we call the legth of the side at height z s then the relationship we get is $\frac{h-z}{h} = \frac{s}{a}$ (note we get $h - z$ and not simply z here since we are measuring the distance from the tip of the triangle down to the height we want). So we can solve this to get $s = \frac{a(h-z)}{h}$ and so the area of the cross section will be given (as a function of z) by $\frac{\sqrt{3}}{4}(\frac{a(h-z)}{h})^2 = \frac{a^2\sqrt{3}(h-z)^2}{4h^2}$. We can now finally write down an integral:

$$\int_0^h \frac{a^2\sqrt{3}(h-z)^2}{4h^2}\,dz = [\frac{-a^2\sqrt{3}(h-z)^3}{12h^2}]_0^h = \frac{a^2h\sqrt{3}}{12}$$