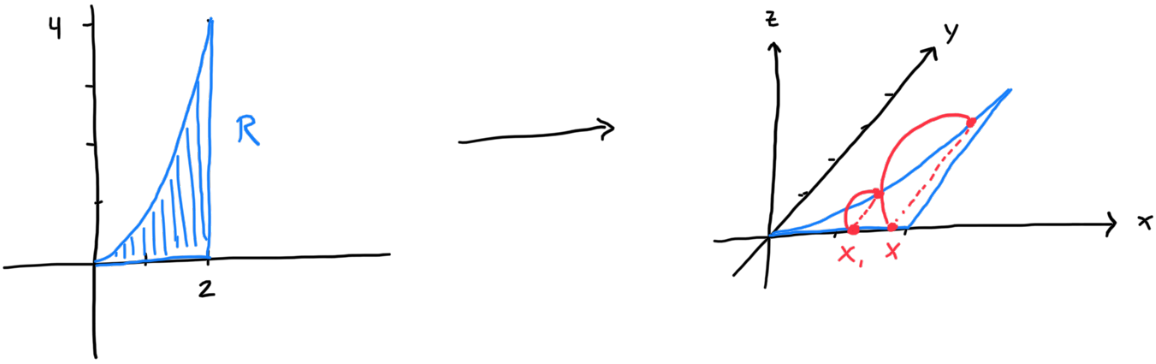
Math 221 Worksheet 26
December 8, 2020
Section 5.2: Volumes

1. Let R denote the region bounded by the curves $y = x^2$, $x = 2$, and $y = 0$. Find the volume of the solid whose horizontal base is R and whose vertical cross sections are semi-disks.



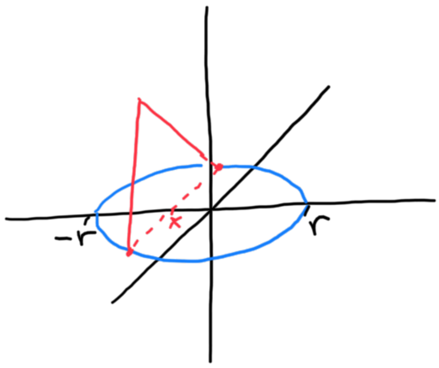
The semi-disk cross section at x has diameter x^2,

so its area is $A(x) = \pi\left(\frac{x^2}{2}\right)^2 = \frac{\pi x^4}{4}$.

But then you need to half this area, since the shape is half a circle.

$$\text{Volume} = \int_0^2 A(x)\,dx = \int_0^2 \frac{\pi x^4}{4}\,dx$$
$$= \frac{\pi x^5}{20}\Big|_0^2 = \boxed{\frac{8\pi}{5}}$$

2. Find the volume of the solid whose horizontal base is a disk of radius r and whose vertical cross sections are equilateral triangles. (Hint: The area of an equilateral triangle with side-length s is $\frac{\sqrt{3}s^2}{4}$).



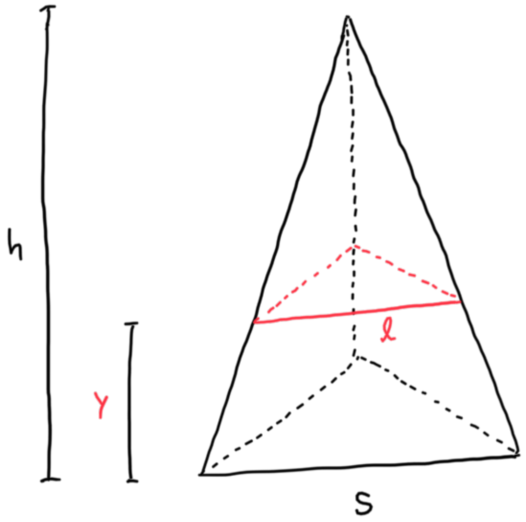
circle: $x^2 + y^2 = r^2$

The equilateral triangle cross section at x has side-length $2\sqrt{r^2 - x^2}$. So its area is $A(x) = \frac{\sqrt{3}}{4} \cdot 4(r^2 - x^2)$

$$= \sqrt{3}(r^2 - x^2).$$

$$\text{Volume} = \int_{-r}^{r} \sqrt{3}(r^2 - x^2)\,dx$$
$$= \sqrt{3}\left(r^2 x - \frac{x^3}{3}\right)\Big|_{-r}^{r} = 2\sqrt{3}\left(r^3 - \frac{r^3}{3}\right) = \boxed{\frac{4r^3}{\sqrt{3}}}$$

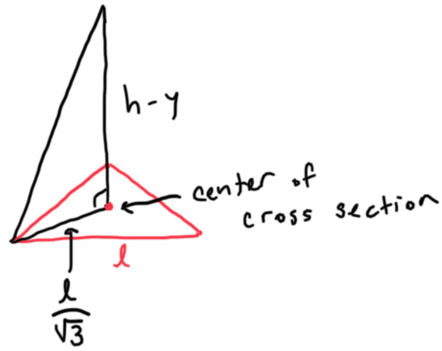
3. Find a formula for the volume of a right pyramid whose base is an equilateral triangle of side-length s and whose height is h.



Let ℓ be the side-length of the red cross section. (Note that it's an equilateral triangle.)

By similar triangles,

$$\frac{\ell}{s} = \frac{h-y}{h}.$$

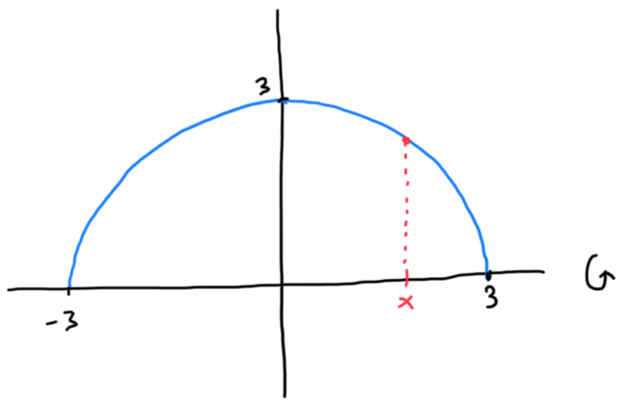


So the area of the cross section at y is $A(y) = \dfrac{\sqrt{3}\, s^2 (h-y)^2}{4h^2}$.

$$\text{Volume} = \int_0^h A(y)\,dy$$

$$= \frac{\sqrt{3}\, s^2}{4h^2} \int_0^h (h-y)^2\,dy = -\frac{\sqrt{3}\, s^2}{4h^2} \cdot \frac{(h-y)^3}{3}\Big|_0^h = \boxed{\frac{\sqrt{3}\, s^2 h}{12}}$$

4. Find the volume of the solid obtained by revolving the region bounded by the curves $y = \sqrt{9 - x^2}$ and $y = 0$ about the x-axis.



The cross section at x is a disk of radius $\sqrt{9 - x^2}$.

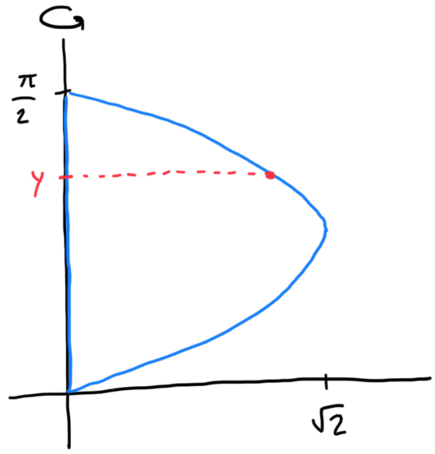
Its area is $A(x) = \pi(9 - x^2)$.

$$\text{Volume} = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} \pi(9 - x^2)\,dx$$

$$= \pi\left(9x - \frac{x^3}{3}\right)\Big|_{-3}^{3}$$

$$= 2\pi\left(27 - \frac{27}{3}\right) = \boxed{36\pi}$$

5. Find the volume of the solid obtained by revolving the region enclosed by the curves $x = \sqrt{2\sin(2y)}$ $(0 \le y \le \frac{\pi}{2})$ and $x = 0$ about the y-axis.

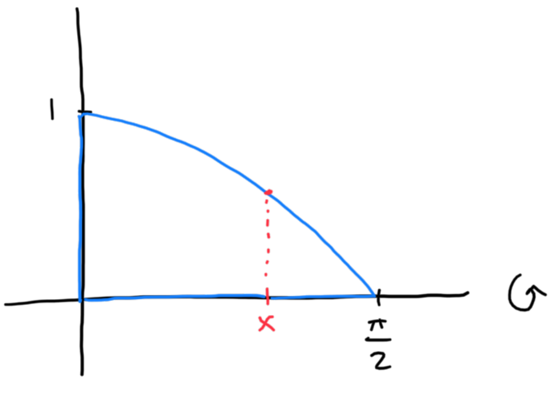


The horizontal cross section at y is a disk of radius $\sqrt{2\sin(2y)}$.

Its area is $A(y) = \pi \cdot 2\sin(2y)$.

$$\begin{aligned}
\text{Volume} &= \int_0^{\frac{\pi}{2}} A(y)\,dy \\
&= \int_0^{\frac{\pi}{2}} 2\pi \sin(2y)\,dy \\
&= -\pi\cos(2y)\Big|_0^{\frac{\pi}{2}} \\
&= -\pi(-1) - (-\pi) \quad = \boxed{2\pi}
\end{aligned}$$

6. Find the volume of the solid obtained by revolving the region bounded by the curves $y = \sqrt{\cos(x)}$ $(0 \le x \le \frac{\pi}{2})$, $y = 0$, and $x = 0$ about the x-axis.



The cross section at x is a disk of radius $\sqrt{\cos x}$.

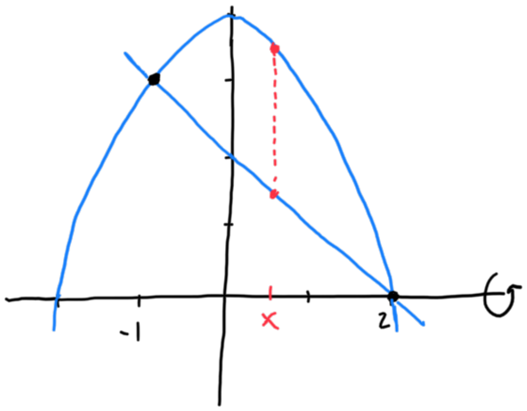
Its area is $A(x) = \pi\cos x$.

$$\text{Volume} = \int_0^{\frac{\pi}{2}} \pi\cos(x)\,dx = \pi\sin x\Big|_0^{\frac{\pi}{2}} = \boxed{\pi}$$

7. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by the curves $y = 4 - x^2$ and $y = 2 - x$ about the x-axis.

$$4 - x^2 = 2 - x \iff x^2 - x - 2 = 0 \iff x = -1, 2$$

The curves intersect at $(-1, 3)$ and $(2, 0)$



The cross section at x is an annulus with big radius $4 - x^2$ and small radius $2 - x$.

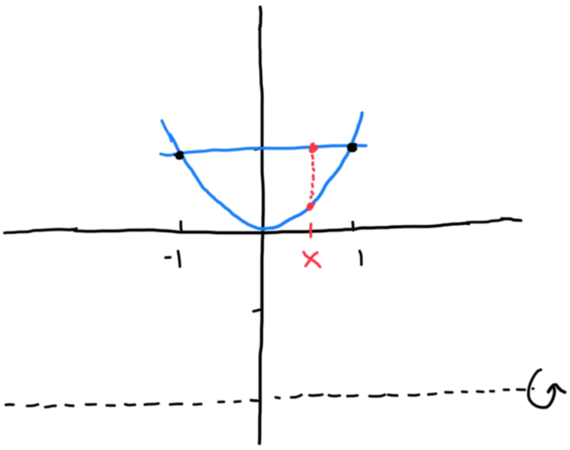
Its area is $A(x) = \pi\left[(4-x^2)^2 - (2-x)^2\right]$.

$$\text{Volume} = \int_{-1}^{2} A(x)\,dx$$

$$= \boxed{\int_{-1}^{2} \pi\left[(4-x^2)^2 - (2-x)^2\right] dx}$$

8. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = 1$ about the line $y = -2$.

Intersections at $(-1, 1)$ and $(1, 1)$.



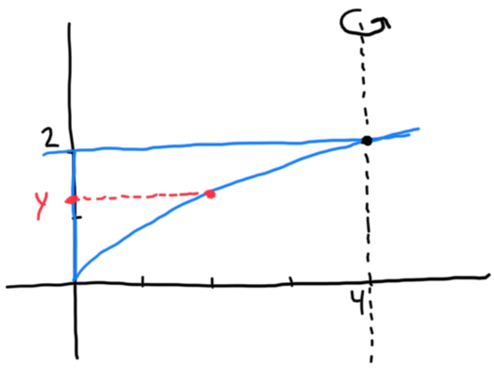
The cross section at x is an annulus with big radius 3 and small radius $x^2 + 2$.

Its area is $A(x) = \pi\left[9 - (x^2+2)^2\right]$.

$$\text{Volume} = \int_{-1}^{1} A(x)\,dx$$

$$= \boxed{\int_{-1}^{1} \pi\left[9 - (x^2+2)^2\right] dx}$$

9. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by the curves $y = \sqrt{x}$, $y = 2$, and $x = 0$ about the line $x = 4$.



The cross section at y is an annulus with big radius 4 and small radius $4 - y^2$.

Its area is $A(y) = \pi\left[16 - (4 - y^2)^2\right]$.

$$\text{Volume} = \int_{0}^{2} A(y)\,dy$$

$$= \boxed{\int_{0}^{2} \pi\left[16 - (4 - y^2)^2\right] dy}$$