Math 221 Worksheet 4
September 15, 2020
Section 1.7 - Formal definition of limit

Formal definition of limit. We write $\lim_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

1. Find a number $\delta > 0$ such that if $|x - 3| < \delta$ then $|3x - 9| < 1$. Find a (possibly different) number $\delta > 0$ such that if $|x - 3| < \delta$ then $|3x - 9| < 0.09$.

$|3x-9| < 1 \iff 3|x-3| < 1$
$\iff |x-3| < \frac{1}{3}$.

So if $\delta = \frac{1}{3}$, then $|x-3| < \delta$ implies $|3x-9| < 1$.

$|3x-9| < 0.09 \iff 3|x-3| < 0.09$
$\iff |x-3| < 0.03$.

So if $\delta = 0.03$, then $|x-3| < \delta$ implies $|3x-9| < 0.09$.

2. Using the formal definition of limit, prove the following:

(a) $\lim_{x \to 3} 3x = 9$

Let $\varepsilon > 0$ be given.

$|3x-9| < \varepsilon \iff 3|x-3| < \varepsilon$
$\iff |x-3| < \frac{\varepsilon}{3}$.

So if $\delta = \frac{\varepsilon}{3}$, then $|x-3| < \delta$ implies $|3x-9| < \varepsilon$.

(b) $\lim_{x \to -3} 6x + 7 = -11$

Let $\varepsilon > 0$ be given.

$|6x+7-(-11)| < \varepsilon \iff 6|x+3| < \varepsilon$
$\iff |x+3| < \frac{\varepsilon}{6}$.

So if $\delta = \frac{\varepsilon}{6}$, then $|x-(-3)| < \delta$ implies $|6x+7-(-11)| < \varepsilon$.

(c) $\lim_{u \to 0} \sin u = 0$ (hint: $|\sin u| \leq |u|$ for all numbers u)

Let $\varepsilon > 0$ be given.

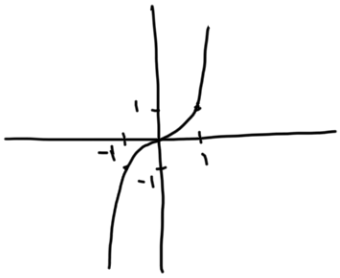
$|\sin u - 0| = |\sin u| \leq |u|$, so if $\delta = \varepsilon$,

then $|u-0| < \delta$ implies $|\sin u - 0| < \varepsilon$.

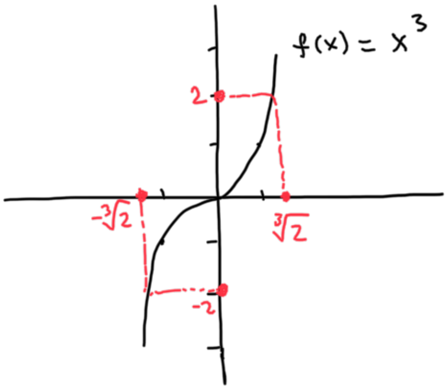
3. Consider the function $f(x) = x^3$.

(a) Determine $\lim_{x \to 0} f(x)$ and sketch the graph of f.

$\lim_{x \to 0} f(x) = 0^3 = 0$ (by limit laws)



(b) Let $\epsilon = 2$ and use your graph from (a) to find a number $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x)| < \epsilon$.



$\delta = \sqrt[3]{2}$ will work

(c) Repeat part (b) with $\epsilon = 1$. If you were to repeat (b) for an *arbitrary* $\varepsilon > 0$, what δ would you choose?

$\varepsilon = 1$: $\delta = \sqrt[3]{1} = 1$ will work.

For arbitrary ε, choose $\delta = \sqrt[3]{\varepsilon}$.

4. Challenge: Prove that $\lim_{x \to 1} x^2 = 1$. (Hint: If $|x - 1| < \delta$, then how large can $|x + 1|$ be?)

Let $\varepsilon > 0$ be given.

$|x^2 - 1| < \varepsilon \iff |x-1||x+1| < \varepsilon$

Assume $|x-1| < \delta$. Then $|x+1| < 2 + \delta$ (because

$|x+1| = |x - 1 + 2| \leq |x-1| + 2 < \delta + 2$).

So $|x-1||x+1| < \delta(2+\delta)$. Therefore we want to choose

δ such that $\delta(2+\delta) < \varepsilon$. Assume $\delta \leq 1$. Then

$\delta(2+\delta) \leq \delta \cdot 3 < \varepsilon$ if $\delta \leq \frac{\varepsilon}{3}$. So $\delta = \min\{1, \frac{\varepsilon}{3}\}$ works.