Math 221 Worksheet 4
September 15, 2020
Section 1.7 - Formal definition of limit

Formal definition of limit. We write $\lim_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

1. Find a number $\delta > 0$ such that if $|x - 3| < \delta$ then $|3x - 9| < 1$. Find a (possibly different) number $\delta > 0$ such that if $|x - 3| < \delta$ then $|3x - 9| < 0.09$.

2. Using the formal definition of limit, prove the following:

   (a) $\lim_{x \to 3} 3x = 9$

   (b) $\lim_{x \to -3} 6x + 7 = -11$

   (c) $\lim_{u \to 0} \sin u = 0$ (hint: $|\sin u| \leq |u|$ for all numbers u)

3. Consider the function $f(x) = x^3$.

   (a) Determine $\lim_{x \to 0} f(x)$ and sketch the graph of f.

   (b) Let $\epsilon = 2$ and use your graph from (a) to find a number $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x)| < \epsilon$.

   (c) Repeat part (b) with $\epsilon = 1$. If you were to repeat (b) for an *arbitrary* $\varepsilon > 0$, what δ would you choose?

4. Challenge: Prove that $\lim_{x \to 1} x^2 = 1$. (Hint: If $|x - 1| < \delta$, then how large can $|x + 1|$ be?)