Math 221 Worksheet 7
September 24, 2020
Section 2.3 - Differentiation Formulas

1. Let $f(x) = x + 2$ and $g(x) = 2x - 1$.

   (a) Compute $f'(x)$ and $g'(x)$.

   $f'(x) = 1$

   $g'(x) = 2$

   (b) Compute $[f(x)g(x)]'$. How does it compare to $f'(x)g'(x)$?

   $f(x)g(x) = (x+2)(2x-1) = 2x^2 + 3x - 2$,

   So $[f(x)g(x)]' = 4x + 3$.

   This is not equal to $f'(x)g'(x)$!

2. Let f and g be functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = -5$, and $g'(2) = 2$. Use differentiation rules to find $h'(2)$ if:

   (a) $h(x) = 3f(x) - g(x)$

   $$h'(2) = 3f'(2) - g'(2) = 3(-1) - 2 = -5$$

   (b) $h(x) = f(x)g(x)$

   $$h'(2) = f'(2)g(2) + f(2)g'(2) = -1(-5) + 3 \cdot 2 = 11$$

   (c) $h(x) = \frac{1}{f(x)}$

   $$h'(2) = \frac{f(2) \cdot 0 - 1 \cdot f'(2)}{f(2)^2} = \frac{-1(-1)}{3^2} = \frac{1}{9}$$

   (d) $h(x) = \frac{g(x)}{f(x)}$

   $$h'(2) = \frac{f(2)g'(2) - g(2)f'(2)}{f(2)^2} = \frac{3 \cdot 2 - (-5)(-1)}{3^2} = \frac{1}{9}$$

3. Compute the derivatives of the following functions:

(a) $f(x) = 4\pi^2$

$$f'(x) = 0$$

(b) $f(x) = x^3 + 2x + 4$

$$f'(x) = 3x^2 + 2$$

(c) $f(x) = \frac{x^2-2x+1}{\sqrt{x}}$

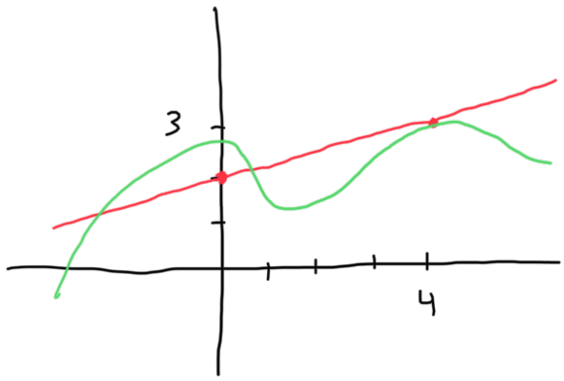
$$f'(x) = \frac{\sqrt{x}\,(2x-2) - (x^2-2x+1)\frac{1}{2\sqrt{x}}}{x}$$

(d) $f(x) = \frac{2x-1}{3x+2}$

$$f'(x) = \frac{(3x+2)\cdot 2 - (2x-1)\cdot 3}{(3x+2)^2}$$

4. Suppose that f is a function whose graph passes through the point $(4, 3)$ and that the tangent line at $(4, 3)$ also passes through the point $(0, 2)$.

(a) Sketch the tangent line along with a *possible* graph of f (make sure to label the two given points).



(b) Find an equation of the tangent line you drew.

slope $= \frac{3-2}{4-0} = \frac{1}{4}$, passes through $(4,3)$.

So its equation is

$$y - 3 = \frac{1}{4}(x-4)$$

(c) What is $f(4)$? What is $f'(4)$?

$f(4) = 3$ because the graph of f passes through $(4,3)$.

$f'(4) = \frac{1}{4}$ because the tangent line at $(4,3)$ has slope $\frac{1}{4}$.

5. Let $f(x) = \frac{x-1}{x+1}$. What is $(x+1)f(x)$? Can you use this to come up with a formula for $f'(x)$ without using the quotient rule?

$(x+1)f(x) = x-1$ for $x \neq -1$ and is undefined at $x=-1$.

Take the derivative of both sides (with $x \neq -1$):

$$1 \cdot f(x) + (x+1)f'(x) = 1$$

Solve for $f'(x)$:

$$f'(x) = \frac{1-f(x)}{x+1} = \frac{1 - \frac{x-1}{x+1}}{x+1}$$

6. Optional/challenge: Let P and Q be polynomials such that $P(1) = Q(1) = 0$ and $Q'(1) \neq 0$. Show that $\lim_{x\to 1} \frac{P(x)}{Q(x)} = \frac{P'(1)}{Q'(1)}$. (If you know L'Hôpital's rule, you may NOT use it!)

Since $P(1) = 0$, we know that $x-1$ is a factor of P, so $P(x) = (x-1)R(x)$ for some polynomial R.

Similarly, $Q(x) = (x-1)S(x)$ for some polynomial S.

So

$$\lim_{x\to 1} \frac{P(x)}{Q(x)} = \lim_{x\to 1} \frac{R(x)}{S(x)}.$$

Now, $P'(x) = 1 \cdot R(x) + (x-1)R'(x)$, so $R(1) = P'(1)$.

Similarly, $S(1) = Q'(1) \neq 0$.

Since $\frac{R}{S}$ is a rational function and $S(1) \neq 0$, $\frac{R}{S}$ is continuous at $x=1$.

So $\lim_{x\to 1} \frac{R(x)}{S(x)} = \frac{R(1)}{S(1)}$.

Altogether, $\lim_{x\to 1} \frac{P(x)}{Q(x)} = \lim_{x\to 1} \frac{R(x)}{S(x)} = \frac{R(1)}{S(1)} = \frac{P'(1)}{Q'(1)}$.