Math 221 Worksheet 9
October 1, 2020
Section 2.6: Implicit Differentiation

1. For each of the following equations find $\frac{dy}{dx}$:

   (a) $x^2 + xy = y^2$

   $2x + 1 \cdot y + xy' = 2yy'$

   $2x + y = y'(2y - x) \rightsquigarrow \boxed{y' = \frac{2x+y}{2y-x}}$

   (b) $\sqrt{xy} = \cos(x + y)$

   $\frac{1}{2\sqrt{xy}}(1 \cdot y + xy') = -\sin(x+y)(1+y')$

   $\frac{y}{2\sqrt{xy}} + \sin(x+y) = y'\left(-\sin(x+y) - \frac{x}{2\sqrt{xy}}\right)$

   $\rightsquigarrow \boxed{y' = \dfrac{\frac{y}{2\sqrt{xy}} + \sin(x+y)}{-\sin(x+y) - \frac{x}{2\sqrt{xy}}}}$

   (c) $\sin(x)\sin(y) = xy^2$

   $\cos(x)\sin(y) + \sin(x)\cos(y)y' = y^2 + x \cdot 2yy'$

   $y'(\sin(x)\cos(y) - 2xy) = y^2 - \cos(x)\sin(y)$

   $\rightsquigarrow \boxed{y' = \frac{y^2 - \cos(x)\sin(y)}{\sin(x)\cos(y) - 2xy}}$

   (d) $\tan(xy^2) = x$

   $\sec^2(xy^2)(y^2 + x \cdot 2yy') = 1$

   $\rightsquigarrow \boxed{y' = \frac{1 - \sec^2(xy^2)y^2}{\sec^2(xy^2) \cdot 2xy}}$

2. The equation $\cos(x^2y) = 3xy^2 + y$ defines a curve. Find the line tangent to it at the point $(0, 1)$.

   Its slope is $\left.\frac{dy}{dx}\right|_{(0,1)}$.

   Implicitly differentiate:

   $-\sin(x^2y)(2xy + x^2y') = 3(y^2 + x \cdot 2yy') + y'$

   Evaluate at $(x,y) = (0,1)$:

   $0 = 3 + y'|_{(0,1)} \rightsquigarrow y'|_{(0,1)} = -3$

   Equation: $\boxed{y - 1 = -3x}$

3. Suppose that f is an invertible function, and let g be its inverse. Suppose additionally that f and g are differentiable, and let $y = f(x)$. What is $g'(y)$?

4. For each of the following equations find $\frac{d^2y}{dx^2}$:

   (a) $xy = x^2 + 1$

   (b) $\sin(y) = xy$

5. The equation $x^2 + y^2 + xy = 1$ defines an ellipse. Among all points (x, y) on this ellipse, which one has the largest y-value and which one has the smallest?

6. The equation $y^2 = x^3 + x + 2$ defines a curve. At which point(s) does it have a vertical tangent line?

7. Let L be the line defined by $4y - 3x = 1$. Find a circle of unit radius that contains the point $(1, 1)$ and whose tangent line at $(1, 1)$ is L.