1. Make an educated guess of the value of the following limits.

   (a) $\lim_{s\to 5} s-3$.

   (b) $\lim_{u\to -2} u^2-\cos(\pi u)$.

   (c) $\lim_{v\to 4} \dfrac{v+3}{4v-2}$.

   Solution. These are all continuous functions. Functions (a) and (b) are continuous everywhere; the functions' graphs never have any jumps or blow-ups. The third function is continuous except where it's not defined when $4v-2=0$, which happens when $v=\frac{1}{2}$. This is not a problem here because we're taking the limit as v moves towards 4. So in each case, the limit is the same as the value of the function with $s=5$, $u=-2$, $v=4$ plugged in, respectively.

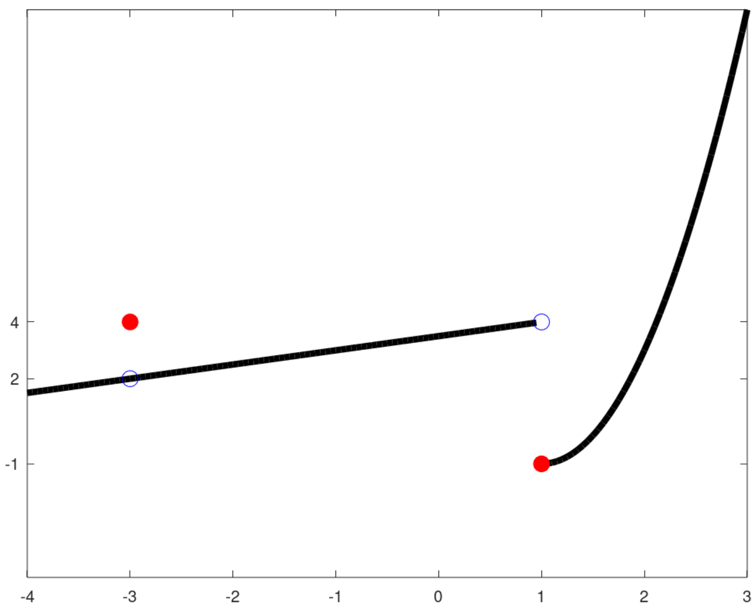
   (a) $\lim_{s\to 5} s-3 = 5-3 = 2$.

   (b) $\lim_{u\to -2} u^2-\cos(\pi u) = (-2)^2-\cos(-2\pi) = 4-1 = 3$.

   (c) $\lim_{v\to 4} \dfrac{v+3}{4v-2} = \dfrac{4+3}{4(4)-2} = \dfrac{7}{14} = \dfrac{1}{2}$.

2. Sketch the graph of an example of a function f that satisfies all of the following: $\lim_{x\to -3^-} f(x)=2$, $\lim_{x\to -3^+} f(x)=2$, $\lim_{x\to 1^-} f(x)=4$, $\lim_{x\to 1^+} f(x)=-1$, $f(-3)=4$, $f(1)=-1$.

   Solution. Here's an example. The blue circles represent holes and the red circles represent function values.



3. Determine the infinite limit.

(a) $\lim_{s\to 1^-} \frac{s^2-4}{s-1}$.

Solution. The numerator approaches $(-1)^2 - 4 = -3$. The denominator approaches 0 from the left-hand side, where the function $g(s) = s - 1$ is negative. Therefore the limit is positive infinity (a negative divided by a negative is positive).

$$\lim_{s\to 1^-} \frac{s^2-4}{s-1} = +\infty.$$

(b) $\lim_{u\to 3^+} \frac{u^2-2u-8}{u^2-6u+9}$.

Solution. Factoring the numerator and denominator is usually a good idea.

$$\lim_{u\to 3^+} \frac{u^2-2u-8}{u^2-6u+9} = \lim_{u\to 3^+} \frac{(u-4)(u+2)}{(u-3)^2}.$$

Now we can see that the numerator approaches $(3-4)(3+2) = -5$ as u moves towards 3 from the right, and the denominator approaches 0 and is always positive (a square is always positive). Hence, the limit is negative infinity (negative divided by positive is negative).

$$\lim_{u\to 3^+} \frac{u^2-2u-8}{u^2-6u+9} = -\infty.$$

(c) $\lim_{t\to 9^-} \frac{\sqrt{t}}{(t-9)^3}$.

Solution. The numerator approaches $\sqrt{9} = 3$ and the denominator approaches 0 through negative values ($t - 9$ is negative to the left of $t = 9$, and therefore so is $(t-9)^3$). So the limit is $-\infty$.

(d) $\lim_{\theta\to \pi^+} \frac{\theta-4}{\sin(\theta)}$

Solution. Similar, the answer is $+\infty$. At $\theta = \pi$, we have $\theta - 4 = \pi - 4 < 0$, and $\sin(\theta)$ is negative when θ is near π on the right-hand side (draw the graph). A negative divided by a negative is positive.

4. Consider the function $f(x) = \frac{2x-3}{(x-2)(x+4)}$.

(a) Find all the vertical asymptotes of f.

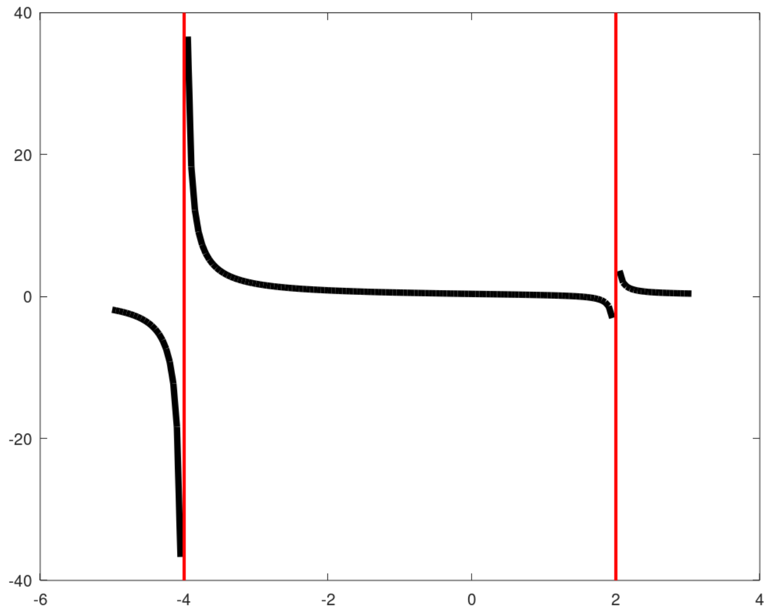
Solution. The denominator is zero when $x = 2$ and $x = -4$. The numerator is zero when $x = \frac{3}{2}$, which does not duplicate any of the denominator's zeros. Therefore there are vertical asymptotes at $x = 2$ and $x = -4$.

(b) Compute $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, $\lim_{x\to -4^+} f(x)$, and $\lim_{x\to -4^-} f(x)$.

Solution. They are $+\infty$, $-\infty$, $+\infty$, $-\infty$, based on analyzing which terms are positive or negative as we move from the left-hand or right-hand side.

(c) Make a rough sketch of the function.

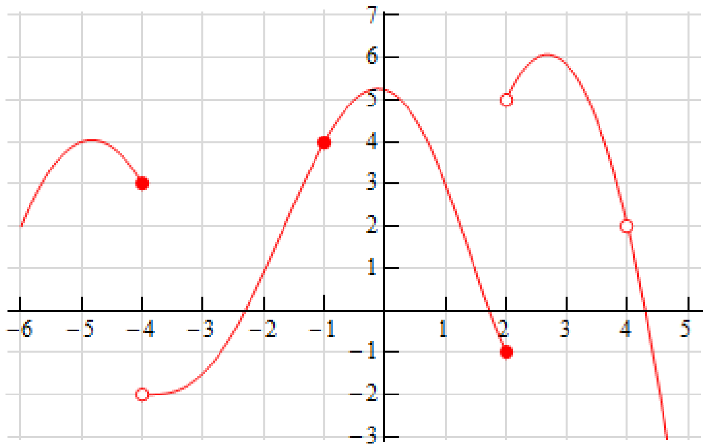
Solution. Red lines represent vertical asymptotes.



5. Consider the functions $f(x) = x+2$, $g(x) = \frac{(x-3)(x+2)}{x-3}$, and $h(x) = \begin{cases} \frac{(x-3)(x+2)}{x-3} & x \neq 3 \\ 8 & x = 3. \end{cases}$. Sketch each of the functions. Then determine the limit as $x \to 3$ of each of the functions. If the limit does not exist, state so.

Solution. $\lim_{x\to 3} f(x) = 3+2 = 5$, since $f(x)$ is continuous everywhere. We find that $g(x)$ simplifies to $g(x) = x+2$ with a hole at $x = 3$. So we can find the limit $\lim_{x\to 3} g(x) = 3+2 = 5$. Likewise, we ignore the jump of $h(x)$ at $x = 3$ and find $\lim_{x\to 3} h(x) = 3+2 = 5$.

6. Below is the graph of $g(t)$. For each of the given points determine the value of $g(a)$, $\lim\limits_{t\to a^-} g(t)$, $\lim\limits_{t\to a^+} g(t)$, and $\lim\limits_{t\to a} g(t)$. If any of the quantities do not exist, explain why.



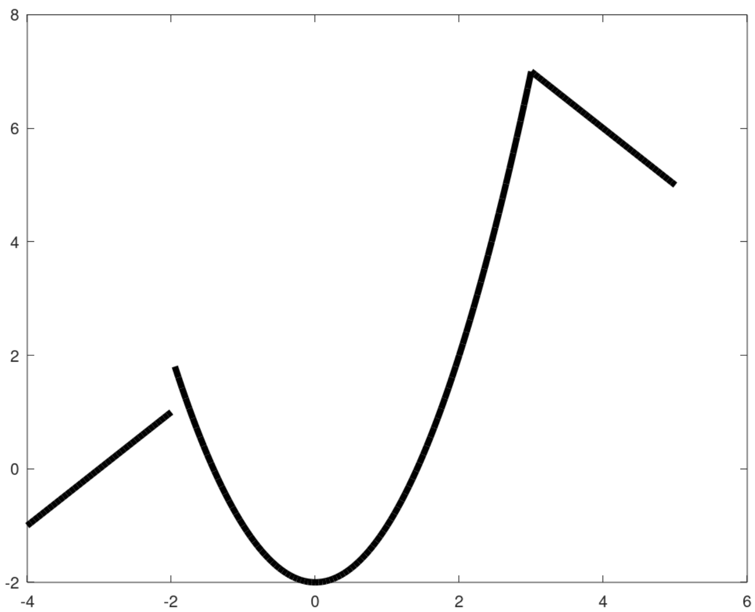
Solution. A limit does not exist when the corresponding right-hand and left-hand limits are not equal.

a	$g(a)$	$\lim_{t\to a-} g(t)$	$\lim_{t\to a+} g(t)$	$\lim_{t\to a} g(t)$
-4	3	3	-2	Does not exist.
-1	4	4	4	4
2	-1	-1	5	Does not exist.
4	Not defined.	2	2	2

7. Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} 3+x, & x < -2 \\ x^2-2, & -2 \le x \le 3 \\ 10-x, & x > 3. \end{cases}$$

Solution.



From the picture, we see that the second and third pieces of the function are glued together seamlessly at $x = 3$, so $\lim_{x\to 3} f(x)$ exists and is equal to 7. The limit of $f(x)$ at $x = -2$ does not exist since there is a jump discontinuity at $x = -2$. The function $f(x)$ is continuous at all other values of x.

8. Consider the function $f(x) = \tan\left(\frac{1}{x}\right)$.

(a) Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$.

Solution. Plugging in these values of x, we have $f(1/\pi) = \tan(\pi) = 0$, $f(1/2\pi) = \tan(2\pi) = 0$, and so on.

(b) Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \ldots$.

Solution. Likewise, $f(4/\pi) = \tan(\frac{1}{4/\pi}) = \tan(\frac{\pi}{4}) = 1$, and so on.

(c) What can you conclude about $\lim_{x \to 0+} \tan\left(\frac{1}{x}\right)$?

Solution. We have found two sequences of numbers approaching $x = 0$ from the right where $f(x)$ approaches two different values, 0 and 1. This means that $\lim_{x\to 0+} \tan(1/x)$ does not exist.