1. Calculate the following using the FTC Part I when appropriate.

   (a) $\dfrac{d}{dx}\left(\int_0^x \sqrt{1-t^2}\,dt\right)$

   Solution. $\sqrt{1-x^2}$, with the restriction $-1 \le x \le 1$.

   (b) $\dfrac{d}{dx}\left(\int_{-5}^x t^3 - 2t^2 + 1 \ dt\right)$

   Solution. $x^3 - 2x^2 + 1$.

   (c) $\dfrac{d}{dx}\left(\int_2^7 t^2\,dt\right)$

   Solution. 0, since $\int_2^7 t^2\,dt$ is a number.

2. In this problem we will use the FTC to evaluate $\dfrac{d}{dx}\left(\int_0^{x^3} t^2 \ dt\right)$

   (a) Let $u(x) = x^3$. Explain why $\frac{d}{dx}F(u(x)) = F'(x^3)\cdot 3x^2$ for any differentiable function F.

   Solution. Chain Rule is why.

   (b) Define $F(x) = \displaystyle\int_0^x t^2 \ dt$. Use FTC to find $F'(x)$.

   Solution. x^2.

   (c) Use the previous two parts to find $\dfrac{d}{dx}\left(\int_0^{x^3} t^2 \ dt\right)$.

   Solution. By the Chain Rule,

   $$\frac{d}{dx}\left(\int_0^{x^3} t^2 \ dt\right) = (x^3)^2(3x^2) = 3x^8.$$

   Writing $F(x) = \int_0^x t^2\,dt$ may help. Then $F'(x) = x^2$ by FTC, so by the Chain Rule:

   $$\frac{d}{dx}F(x^3) = F'(x^3)(3x^2) = 3x^8.$$

3. Compute $\dfrac{d}{dx}\left(\int_2^{1/x} \arctan t \ dt\right)$.

Solution. Similarly to the previous one, from the Chain Rule,

$$\frac{d}{dx}\left(\int_2^{1/x} \arctan t \ dt\right) = \arctan(1/x)\cdot(-\frac{1}{x^2}).$$

4. Let $F(x) = \displaystyle\int_2^x \frac{\cos(\sin(t^2))}{t^3}\, dt$. Compute $F'(x)$.

Solution. Fundamental Theorem of Calculus.

$$F'(x) = \frac{\cos(\sin(x^2))}{x^3},$$

when $x \neq 0$.

5. Compute $\left(\displaystyle\int_{\frac{1}{x^2}}^0 \cos(t^4 \sin(t))\, dt\right)'$.

Solution. From the Chain Rule:

$$\begin{aligned}
\left(\int_{\frac{1}{x^2}}^0 \cos(t^4 \sin(t))\, dt\right)' &= \frac{d}{dx}\left(-\int_0^{\frac{1}{x^2}} \cos(t^4 \sin(t))\, dt\right) \\
&= -\cos\left(\left(\frac{1}{x^2}\right)^4 \sin\left(\frac{1}{x^2}\right)\right)\cdot(-2)x^{-3} \\
&= \frac{2}{x^3}\cos\left(\left(\frac{1}{x^2}\right)\sin\left(\frac{1}{x^2}\right)\right).
\end{aligned}$$

6. Compute $\dfrac{d}{dx}\displaystyle\int_{\pi x}^{\cos(x)} \frac{1}{1+t^3}\, dt$.

Solution. Split up the integral in order to apply FTC / Chain Rule.

$$\begin{aligned}
\frac{d}{dx}\int_{\pi x}^{\cos(x)} \frac{1}{1+t^3}\, dt &= \frac{d}{dx}\left(\int_{\pi x}^0 \frac{1}{1+t^3}\, dt + \int_0^{\cos(x)} \frac{1}{1+t^3}\, dt\right) \\
&= \frac{d}{dx}\left(-\int_0^{\pi x} \frac{1}{1+t^3}\, dt + \int_0^{\cos(x)} \frac{1}{1+t^3}\, dt\right) \\
&= -\frac{1}{1+(\pi x)^3}\cdot\pi + \frac{1}{1+\cos^3(x)}(-\sin(x)) \\
&= \frac{-\pi}{1+\pi^3 x^3} - \frac{\sin(x)}{1+\cos^3(x)}.
\end{aligned}$$

7. On what interval is the function $F(x) = \int_2^x \frac{1}{1+t+t^2}\,dt$ concave up? Concave down? Find the x-coordinates of any inflection points.

Solution. Let's see. By FTC:

$$F'(x) = \frac{1}{1+x+x^2}.$$

Then

$$F''(x) = \frac{-(2x+1)}{(1+x+x^2)^2}.$$

The denominator $(1+x+x^2)^2$ is always positive. Therefore $F''(x) > 0$ when $-(2x+1) > 0$, which happens when $2x+1 > 0$, which happens when $x > \frac{-1}{2}$. The function $F(x)$ is concave up on the interval $(\frac{-1}{2}, +\infty)$. Likewise, $-(2x+1) < 0$ when $x < \frac{-1}{2}$. The function $f(x)$ is concave down on the interval $(-\infty, \frac{-1}{2})$.

8. Let $f(x) = \frac{1}{2}x - 1$.

(a) Sketch the graph of $f(x)$ on the interval $[0, 3]$. Use your picture to calculate the area under the curve on this interval.

Solution. Picture omitted. Signed area is $\frac{-3}{4}$.

(b) Find a function $F(x)$ so that $F'(x) = f(x)$ ($F(x)$ is an antiderivative of $f(x)$).

Solution. $F(x) = \frac{1}{4}x^2 - x$.

(c) Calculate $F(3) - F(0)$.

Solution. $\frac{-3}{4}$. Behold, it is the same number as part (a).

9. Evaluate the following definite integrals.

(a) $\int_1^4 2x^4 - 3x^2\,dx$

Solution. By the Fundamental Theorem of Calculus:

$$\begin{aligned}\int_1^4 2x^4 - 3x^2\,dx &= \int_1^4 \frac{d}{dx}\left(\frac{2}{5}x^5 - x^3\right)dx \\ &= \left(\frac{2}{5}x^5 - x^3\right)\Big|_{x=4} - \left(\frac{2}{5}x^5 - x^3\right)\Big|_{x=1} \\ &\cong 346.2.\end{aligned}$$

(b) $\int_0^4 x\sqrt{x^3}\,dx$

Solution. By the Fundamental Theorem of Calculus:

$$\begin{aligned}\int_0^4 x\sqrt{x^3}\,dx &= \int_0^4 x^{5/2}\,dx \\ &= \int_0^4 \frac{d}{dx}\left(\frac{2}{7}x^{7/2}\right)dx \\ &= \left(\frac{2}{7}x^{7/2}\right)\Big|_{x=4} - \left(\frac{2}{7}x^{7/2}\right)\Big|_{x=1} \\ &= \frac{2}{7}\left(2^7 - 1\right).\end{aligned}$$

(c) $\int_0^{\frac{\pi}{4}} \sin(x)\,dx$

Solution.

$$\int_0^{\frac{\pi}{4}} \sin(x)\,dx = -\cos(\frac{\pi}{4}) - (-\cos(0)) = 1 - \frac{1}{\sqrt{2}}.$$

(d) $\int_0^1 (x^3 - 1)^2\,dx$

Solution. Expand first, then FTC. $(x^3 - 1)^2 = x^6 - 2x^3 + 1$. Therefore:

$$\int_0^1 (x^3-1)^2\,dx = \int_0^1 x^6 - 2x^3 + 1\,dx = \left(\frac{1}{7}x^7 - \frac{1}{2}x^4 + x\right)\Big|_{x=0}^{x=1} = \frac{9}{14}.$$

(e) $\int_{-1}^3 |x-2|\,dx$

Solution. $|x-2| = x-2$ when $x \geq 2$ and $|x-2| = -(x-2)$ when $x < 2$. Split up the integral at $x = 2$, the point where the piecewise function changes formulas.

$$\begin{aligned}\int_{-1}^3 |x-2|\,dx &= \int_{-1}^2 |x-2|\,dx + \int_2^3 |x-2|\,dx \\ &= \int_{-1}^2 -(x-2)\,dx + \int_2^3 x - 2\,dx \\ &= 4.5 + 0.5 \\ &= 5.\end{aligned}$$

(f) $\displaystyle\int_{-1}^{5} (1+3x)\,dx$

Solution.

$$\int_{-1}^{5} (1+3x)\,dx = \left(x + \frac{3}{2}x^2\right)\Big|_{x=-1}^{x=5} = 42.$$

(g) $\displaystyle\int_{0}^{2} (2-x^2)\,dx$

Solution.

$$\int_{0}^{2} (2-x^2)\,dx = \left(2x - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=2} = \frac{-2}{3}.$$

10. Compute $\displaystyle\int_{-1}^{1} x + x^3\,dx$. Does your answer make sense geometrically?

Solution.

$$\int_{-1}^{1} x + x^3\,dx = \left(\frac{1}{2}x^2 + \frac{1}{4}x^4\right)\Big|_{x=-1}^{1} = 0.$$

The geometry is: $f(x) = x + x^3$ is an odd function, and the interval $[-1,1]$ is symmetric about 0. The area between $x = 0$ and $x = 1$ is the exact negative of the area between $x = -1$ and $x = 0$, and the two cancel.

11. Find $\displaystyle\int_{0}^{5} f(x)\ dx$ if

$$f(x) = \begin{cases} 3, & x < 3 \\ x, & x \geq 3 \end{cases}$$

Solution. Split up at $x = 3$.

$$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = \int_0^3 3\,dx + \int_3^5 x\,dx = 9 + 8 = 17.$$

12. Find the area under the curve $y = \sqrt{4x+4}$ and above the x-axis between the vertical lines $x = 0$ and $x = 2$. Sketch a graph of the curve and the area.

Solution. Sketch omitted, but here's how you can sketch it. Write

$$y = \sqrt{4x+4} = \sqrt{4(x+1)} = 2\sqrt{x+1}.$$

So, shift the graph of $y = \sqrt{x+1}$ horizontally by one unit, then scale vertically $2\times$.

13. Find two functions $F_1(x), F_2(x)$ such that $F_1'(x) = F_2'(x) = 4x - \cos(x)$. Use both of them to compute $\displaystyle\int_0^{\pi} 4x - \cos(x)\,dx$. Do you get the same answer?

Solution. The point is, in FTC # 2, you can use any antiderivative $F(x)$ of a given function $f(x)$, because all antiderivatives are constant shifts of one another.

$$\int_a^b f(x)\,dx = F(b) - F(a) = (F(b) + C) - (F(a) + C).$$

14. The velocity (in meters per second) of a particle moving along a line is given by $v(t) = t^2 - 2t - 3$ for $2 \leq t \leq 4$. Find the displacement of the particle and the distance traveled by the particle during the given time interval.

Solution. Displacement is:

$$\int_0^4 v(t)\,dt = \frac{1}{3}t^3 - t^2 - 3t\Big|_{t=0}^{t=4} = \frac{-2}{3}\text{ m.}$$

Distance traveled is:

$$\int_0^4 |v(t)|\,dt.$$

So we need to analyze where $v(t)$ is positive and negative in order to compute the integral.

We factor $v(t) = (t-3)(t+1)$. Using test points, we find that $v(t) < 0$ on the time interval $(-1, 3)$ and $v(t) > 0$ on the time interval $(3, +\infty)$. Therefore, the distance traveled is:

$$\begin{aligned}
\int_0^4 |v(t)|\,dt &= \int_0^3 |v(t)|\,dt + \int_3^4 |v(t)|\,dt \\
&= \int_0^3 -v(t)\,dt + \int_3^4 v(t)\,dt \\
&= -\int_0^3 v(t)\,dt + \int_3^4 v(t)\,dt \\
&= -\left(\frac{1}{3}t^3 - t^2 - 3t\right)\Big|_{t=0}^{t=3} + \left(\frac{1}{3}t^3 - t^2 - 3t\right)\Big|_{t=3}^{t=4} \\
&= \frac{34}{3}\text{ m.}
\end{aligned}$$

Here, we are using the fact that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$. Therefore $|v(t)| = v(t)$ when $v(t) > 0$ and $|v(t)| = -v(t)$ when $v(t) < 0$. We split the integral along the points where $v(t)$ changes sign.