1. Find the general indefinite integrals.

(a) $\int x\sqrt{x}\,dx$

Solution.

$$\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2} + C.$$

(b) $\int (1-t)(2+t^2)\,dt$

Solution.

$$\int (1-t)(2+t^2)\,dt = \int (-t^3 + t^2 - 2t + 2)\,dt = \frac{-t^4}{4} + \frac{t^3}{3} - t^2 + 2t + C.$$

(c) $\int 13 + 2t^4\,dt$

Solution.

$$\int 13 + 2t^4\,dt = 13t + \frac{2}{5}t^5 + C.$$

(d) $\int \cos(u+3)\,du$

Solution.

$$\int \cos(u+3)\,du = \sin(u+3) + C.$$

Notice how antiderivatives work smoothly when the variable u is changed to $u+3$.

(e) $\int (s+3)^2\,ds$

Solution.

$$\int (s+3)^2\,ds = \frac{1}{3}(s+3)^3 + C.$$

You can also expand $(s+3)^2 = s^2 + 6s + 9$ and integrate that way. You will see that the two answers agree.

(f) $\int \frac{3x^3 + 5x^2 - 1}{3x^2}\,dx$

Solution.

$$\int \frac{3x^3 + 5x^2 - 1}{3x^2}\,dx = \int (x + \frac{5}{3} - \frac{1}{3}x^{-2}\,dx = \frac{1}{2}x^2 + \frac{5}{3}x + \frac{1}{3}x^{-1} + C.$$

(g) $\int \sqrt{t} - \frac{2}{\sqrt{t}}\,dt$

Solution.

$$\int \sqrt{t} - \frac{2}{\sqrt{t}}\,dt = \int (t^{1/2} - 2t^{-1/2})\,dt = \frac{2}{3}t^{3/2} - 4t^{1/2} + C.$$

(h) $\int (x^2+1)^2\,dx$

Solution.

$$\int (x^2+1)^2\,dx = \int (x^4+2x^2+1)\,dx = \frac{1}{5}x^5 + \frac{2}{3}x^3 + x + C.$$

2. Water flows from the bottom of a storage tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$. Find the amount of water that flows out of the tank during the first ten minutes.

Solution. This amount of water is $\int_0^{10} r(t)\,dt$, which is:

$$\int_0^{10} (200-4t)\,dt = \big(200t - 2t^2\big)\Big|_{t=0}^{t=10} = 1800 \text{ L}.$$

3. If $w'(t)$ is the Dane County's rate of population growth (unit people per year, with time $t = 0$ representing the year 2000), what does $\displaystyle\int_5^{10} w'(t)\,dt$ represent?

Solution. $\int_5^{10} w'(t)\,dt$ is the accumulation of people between the years 2005 and 2010, according to our time scale. In other words, $\int_5^{10} w'(t)\,dt$ is the change of population of Dane County between 2005 and 2010. This can be positive or negative, depending on whether more people have come to or left Dane County.

4. A honeybee population starts with 100 bees and increases at a rate of $n'(t)$ bees per week. What does $100 + \displaystyle\int_0^{15} n'(t)\,dt$ represent?

Solution. This represents the honeybee population 15 weeks after the start $t = 0$. Because:

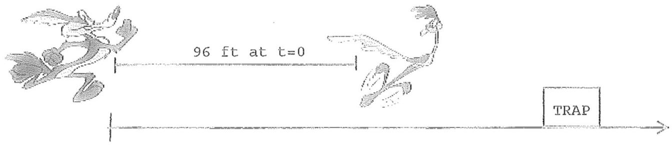
$$\begin{aligned} 100 + \int_0^{15} n'(t)\,dt &= \text{(initial pop.)} \ + \ (\text{ change in pop. from } t=0 \text{ to } t=5\ ) \\ &= (\text{ final pop. }) \end{aligned}$$

5. A particle is moving with an acceleration of $a(t) = 2t + 5$ m/s^2. The initial velocity of the particle is $v(0) = 4$, with $0 \le t \le 10$. Find the velocity at time t, as well as the total distance traveled.

Partial solution. Antidifferentiate to find velocity. $v(t) = t^2 + 5t + C$. Using the equation $v(0) = 4$ gives $C = 4$. So $v(t) = t^2 + 5t + 4$. On the time interval $0 \le t \le 10$, we see $v(t) \ge 0$ (either by factoring and looking at test points, or just noticing all the coefficients of t are positive). Therefore the total distance traveled is:

$$\int_0^{10} |v(t)|\,dt = \int_0^{10} v(t)\,dt = \frac{1870}{3}.$$

6. A coyote is chasing a roadrunner in a straight line. Suppose the roadrunner has a head start of 96 ft. Suppose the coyote is running at a speed of $4t$ ft/sec, and that the road runner is running at a speed of 4 ft/sec.



(a) Has the coyote caught up with the roadrunner after 7 seconds?

(b) Has the coyote caught up with the roadrunner after 8 seconds?

(c) The roadrunner has set a trap for the coyote 120 ft from the coyote's starting point. Will the roadrunner be able to escape from the coyote? That is, will the coyote get to the trap before he catches the roadrunner?

Solution. The coyote's velocity is $c'(t) = 4t$, so $c(t) = 2t^2 + C_0$. Choosing coordinates, we can set the coyote's position at $x = 0$ at time $t = 0$, so $c(t) = 2t^2$. The roadrunner's velocity is $r'(t) = 4$, so $r(t) = 4t + C_1$. The initial condition says $r(t) = 4t + 96$, since we need $r(0) = 96$ with our choice of coordinates.

Let's find when the coyote catches up to the roadrunner. This would correspond to the equation $c(t) = r(t)$. So:

$$\begin{aligned} 2t^2 &= 4t + 96 \\ \Rightarrow t^2 - 2t - 48 &= 0 \\ \Rightarrow (t-8)(t-6) &= 0 \\ \Rightarrow t = 6 &\text{ or } t = 8. \end{aligned}$$

So by time $t = 6$ seconds, the coyote has caught up to the roadrunner. The answers to (a) and (b) are "yes."

Let's find out at what position the coyote catches up to the roadrunner. This would simply be the location $x = c(6) = 72$ ft. The coyote catches up to the roadrunner 72 feet after the coyote's starting position, which is before 120 ft. from the starting position. In other words, the coyote catches the roadrunner before the coyote falls in the trap.

7. Try to find an antiderivative for each of the following functions. For exactly two of the functions listed, it will not be possible to find an explicit formula. (Although, we can always write the formula $F(x) = \int_0^x f(t)\,dt$, which gives an antiderivative of $f(x)$ by FTC Part 1.)

Solutions. In each case we'll call $F(x)$ the antiderivative of $f(x)$. We can always check our work by checking that $F'(x) = f(x)$.

(a) $f(x) = \sin(2x)$

Solution. $F(x) = \frac{1}{2}\sin(2x)$.

(b) $f(x) = \sin(x + 4)$

Solution. $F(x) = -\cos(x + 4)$.

(c) $f(x) = \sin(x^2)$

Solution. Can't find explicit formula.

(d) $f(x) = 2x\sin(x^2)$

Solution. $F(x) = -\cos(x^2)$.

(e) $f(x) = (x-1)$

Solution. $F(x) = \frac{1}{2}(x-1)^2$. Or we could simply antidifferentiate term-by-term can find $F(x) = \frac{1}{2}x^2 - x$. These are different antiderivatives; they differ by a constant $+1$.

(f) $f(x) = (x-1)^2$

Solution. $F(x) = \frac{1}{3}(x-1)^3$.

(g) $f(x) = \sqrt{x-1}$

Solution. $F(x) = \frac{2}{3}(x-1)^{3/2}$.

(h) $f(x) = \sqrt{x^2-1}$

Solution. Can't find explicit formula. There is no way we could do u-subsitution, which is our only method for antidifferentiating besides noticing linear changes of variables.

8. Use a substitution to compute the following integrals.

Solutions. In each case we use the formula:

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(t)\,dt.$$

That is what's going on behind the scenes with u-substitution. We spot a function $u(x)$ inside another function, with the derivative $u'(x)$ multiplied on the outside of the function f. We would normally write the second integral above as $\int_{u(a)}^{u(b)} f(u)\,du$, where now we think of u as a variable, and $u(a)$ and $u(b)$ are the new bounds of integration given by plugging in the old integration bounds into $u(x)$.

(a) $\displaystyle\int_0^{\pi/3} \sin(2x)\,dx$

Solution. Take $u = 2x$. Then $du = 2\,dx$. Plugging in $x = 0$ and $x = \pi/3$ into u gives new integration bounds $u = 0$ and $u = 2\pi/3$. So:

$$\begin{aligned}\int_0^{\pi/3} \sin(2x)\,dx &= \int_0^{\pi/3} \frac{1}{2}\sin(2x)\,2\,dx \\ &= \int_0^{2\pi/3} \frac{1}{2}\sin(u)\,du \\ &= \left(\frac{-1}{2}\cos(u)\right)\Bigg|_{u=0}^{u=2\pi/3} \\ &= \frac{-1}{2}\left(\frac{-1}{2} - 1\right) \\ &= \frac{3}{4}.\end{aligned}$$

Look at what we did here carefully. (1) We replaced all $2x$'s by u's. (2) We rewrote our integrand $\sin(2x)$ by multiplying by $\frac{1}{2}\cdot 2$, to ensure that the full term $2\,dx$ appeared in our integrand. This was necessary in order to make a du appear. Always, your algebra should be written very neatly so you can correctly rewrite the integrand to make the full du term appear, and so that you can be sure all u terms are replaced carefully. Notice that we cleaned up all the x's; after our substitution, there were no x's left over. You can't force this; you just have to write the algebra carefully to make sure it is supposed to happen, like in our work here. Don't break these rules! Sloppy work these problems will not earn you any credit on exams and will almost always give you incorrect answers. Do many practice problems so you can see the variety of ways we rewrite integrals to allow proper u-substitutions.

(b) $\int_{-\pi}^{0} \sin(x+4)\,dx$

Solution. Take $u = x + 4$, then $du = dx$. The change of variables gives the bound of integration $u = 4 - \pi$ to 4. So:

$$\int_{-\pi}^{0} \sin(x+4)\,dx = \int_{4-\pi}^{4} \sin(u)\,du = -\cos(4) + \cos(4-\pi).$$

(c) $\int_{0}^{\sqrt{\pi}} 2x\sin(x^2)\,dx$

Solution. Take $u = x^2$. Then $du = 2x\,dx$, and we have integration bounds $u = 0$ to $u = \pi$. So:

$$\int_{0}^{\sqrt{\pi}} 2x\sin(x^2)\,dx = \int_{0}^{\pi} \sin(u)\,du = -\cos(\pi) + \cos(0) = 2.$$

(d) $\int_{-17}^{-3} (x-1)\,dx$

Solution. No substitution necessary, although you can use substitution if you want.

$$\int_{-17}^{-3} (x-1)\,dx = \left(\frac{1}{2}x^2 - x\right)\Bigg|_{x=-17}^{x=-3} = -154.$$

(e) $\int (x-1)^2\,dx$

Solution. More than one way to do this. We u-substitute $u = x - 1$. Then $du = dx$ so:

$$\int (x-1)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x-1)^3 + C.$$

(f) $\int \sqrt{x-1}\,dx$

No solution. Do yourself carefully. The answer is $\frac{2}{3}(x-1)^{\frac{3}{2}} + C$.

(g) $\int_{1}^{3} x\sqrt{x^2-1}\,dx$

Solution. u-sub $u = x^2 - 1$. Then $du = 2x\,dx$, and the bounds of integration are $u = 0$ and $u = 8$.

$$\begin{aligned}
\int_{1}^{3} x\sqrt{x^2-1}\,dx &= \int_{1}^{3} x\sqrt{x^2-1}\cdot\frac{1}{2}\cdot 2\,dx \\
&= \int_{0}^{8} \frac{1}{2}u^{1/2}\,du \\
&= \left(\frac{1}{3}u^{3/2}\right)\Bigg|_{u=0}^{u=8} \\
&= \frac{8^{3/2}}{3}.
\end{aligned}$$

(h) $\displaystyle\int t\sec(3t^2)\tan(3t^2)\,dt$

Solution. u-substitute $u = 3t^2$. Then $du = 6t$. So:

$$\begin{aligned}\int t\sec(3t^2)\tan(3t^2)\,dt &= \int \frac{1}{6}\sec(3t^2)\tan(3t^2)\,6t\,dt \\ &= \int \frac{1}{6}\sec(u)\tan(u)\,du \\ &= \frac{1}{6}\sec(u) + C \\ &= \frac{1}{6}\sec(3t^2) + C.\end{aligned}$$

(i) $\displaystyle\int_0^{2\pi} |\sin(t)|\,dt$

Solution. We know $\sin(t) = \sin(t)$ when $\sin(t) \geq 0$ and $\sin(t) = -\sin(t)$ when $\sin(t) < 0$. So we have to split up the interval of integration along where $\sin(t)$ changes sign, namely at $t = 0$ and $t = \pi$.

$$\begin{aligned}\int_0^{2\pi} |\sin(t)|\,dt &= \int_0^{\pi} |\sin(t)|\,dt + \int_{\pi}^{2\pi} |\sin(t)|\,dt \\ &= \int_0^{\pi} \sin(t)\,dt + \int_{\pi}^{2\pi} (-\sin(t))\,dt \\ &= (-\cos(\pi) + \cos(0)) + (\cos(2\pi) - \cos(\pi)) \\ &= 4.\end{aligned}$$

(j) $\displaystyle\int \tan^3(\theta)\sec(\theta)\,d\theta$

Solution. This is very hard. Breaking off a power of $\tan\theta$ and using the Pythagorean identity $\tan^2\theta + 1 = \sec^2\theta$, we obtain:

$$\begin{aligned}\int \tan^3(\theta)\sec(\theta)\,d\theta &= \int \tan^2\theta \cdot \sec\theta\tan\theta\,d\theta \\ &= \int (\sec^2\theta - 1)\cdot\sec\theta\tan\theta\,d\theta \\ &= \int (u^2 - 1)\,du \\ &= \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\sec^3(\theta) - \sec(\theta) + C.\end{aligned}$$

We made the u-substitution $u = \sec(\theta)$.