Logarithms.

1. Compute the derivatives of the following functions.

   (a) $f(x) = \ln(3x^2 - 5x)$

   Solution.

$$f'(x) = \frac{6x - 5}{3x^2 - 5x}.$$

   (b) $g(u) = \dfrac{u + \ln(5u)}{\sin(u)}$.

   Solution. I will rewrite slightly

$$g(u) = \frac{u + \ln(5) + \ln(u)}{\sin(u)}.$$

   Then by the quotient rule:

$$g'(u) = \frac{(1 + \frac{1}{u})\sin(u) - (u + \ln(5) + \ln(u))\cos(u)}{\sin^2(u)}.$$

   (c) $f(s) = \ln\left(\sqrt{\frac{2s+1}{4s}}\right)$.

   Solution. Logarithm identities can make our life easier.

$$\begin{aligned}
f(s) &= \ln\left(\sqrt{\frac{2s+1}{4s}}\right) \\
\Rightarrow f(s) &= \ln\left(\left(\frac{2s+1}{4s}\right)^{1/2}\right) \\
\Rightarrow f(s) &= \frac{1}{2}\ln\left(\frac{2s+1}{4s}\right) \\
\Rightarrow f(s) &= \frac{1}{2}\left(\ln(2s+1) - \ln(4s)\right) \\
\Rightarrow f'(s) &= \frac{1}{2}\left(\frac{2}{2s+1} - \frac{4}{4s}\right) \\
\Rightarrow f'(s) &= \frac{1}{2}\left(\frac{2}{2s+1} - \frac{1}{s}\right).
\end{aligned}$$

(d) $h(u) = e^{4u}\ln(ue^u)$

Solution. I'll rewrite $h(u) = e^{4u}(\ln(u) + \ln(e^u)) = e^{4u}(\ln(u) + u)$. Then:

$$h'(u) = 4e^{4u}(\ln(u) + u) + e^{4u}(\frac{1}{u} + 1).$$

(e) $y = x\log_4(\sin(x))$

Solution. Rewrite with log identity $\log_b(x) = \frac{\ln(x)}{\ln(b)}$. Then $y = \frac{1}{\ln(4)} \cdot \ln(\sin(x))$. Then:

$$y' = \frac{1}{\ln(4)} \cdot \frac{1}{\sin(x)} \cdot \cos(x) = \frac{1}{\ln(4)}\cot(x).$$

(f) $y = \log_2(x\log_5 x)$

Solution. Rewrite:

$$y = \frac{1}{\ln(2)}\ln\left(\frac{1}{\ln(5)}x\ln(x)\right) = \frac{1}{\ln(2)}\left(\ln(x) + \ln(\ln(x)) - \ln(5)\right).$$

Then:

$$y' = \frac{1}{\ln(2)}\left(\frac{1}{x} + \frac{1}{x\ln(x)}\right).$$

2. Find the equation of the tangent line to the curve $y = \ln(x^2)$ when $x = e$.

Solution. Using the log identity $y = 2\ln(x)$, we see easily $y' = \frac{2}{x}$. So $y'(e) = \frac{2}{e}$. We compute $y(e) = 2\log(e) = 2$. Therefore, the tangent line is:

$$y = \frac{2}{e}(x - e) + 2.$$

3. Sketch the graph of $f(x) = x + e^x$ using the curve sketching techniques you learned in Chapter 3.

Solution. We compute $f'(x) = 1 + e^x$. Since $e^x > 0$ always, then $f'(x) > 0$ always. And $f''(x) = e^x > 0$ always also. The function is always increasing and concave up. As $x \to -\infty$, we have $e^x \to 0$ and $x \to -\infty$. Therefore $f(x) \to -\infty$ as $x \to -\infty$. And since x and e^x both diverge to $+\infty$ as $x \to +\infty$, then $f(x) \to +\infty$ as $x \to +\infty$.

I don't know where the graph crosses the x-axis, but it will be somewhere where $x < 0$, since $f(0) = 1$. In fact, it must be between -1 and 0, since $f(-1) = -1 + \frac{1}{e} < 0$.

Graph yourself: `https://www.desmos.com/calculator`.

4. Find y' if $2e^y + \ln(xy) = 2x^2y + 4$.

Solution. Implicit diff.

$$2e^y y' + \frac{y + xy'}{xy} = 4xy + 2x^2 y'.$$

Solving for y' gives

$$y' = \frac{4xy - \frac{1}{x}}{2e^y - 2x^2 + \frac{1}{y}}.$$

5. Find a formula for the $n-$th derivative of $g(s) = e^{4s}$.

Solution. We compute $g'(s) = 4e^{4x}$, then $g''(s) = 4^2e^{4x}$, then $g^{(3)}(x) = 4^3e^{4x}$, and so on. After ruminating for a while, we realize the power of 4 will always be identical to the number of derivatives we have taken. We conclude:

$$g^{(n)}(s) = 4^n e^{4s}.$$

6. Compute the following integrals.

(a) $\displaystyle\int_0^{\frac{e-1}{2}} \frac{5}{1+2x}\,dx$

Solution. We can see $\frac{5}{2}\ln|1+2x|$ is an antiderivative. (Linear u-substitution.) Therefore:

$$\int_0^{\frac{e-1}{2}} \frac{5}{1+2x}\,dx = \left(\frac{5}{2}\ln|1+2x|\right)\Big|_{x=0}^{x=(e-1)/2} = \frac{5}{2}.$$

(b) $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

Solution. u-sub $u = \ln(x)$.

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(u)\,du = -\cos(u) + C = -\cos(\ln(x)) + C.$$

(c) $\displaystyle\int_1^e \frac{(\ln t)^4}{t} dt$

Solution. Let $u(t) = \ln(t)$. Then $du = \frac{1}{t}\,dt$. The endpoints change to $u(1) = 0$ and $u(e) = 1$. So:

$$\int_1^e \frac{(\ln t)^4}{t} dt = \int_0^1 u^4\,du = \frac{1}{5}u^5\Big|_{u=0}^{u=1} = \frac{1}{5}.$$

(d) $\int \frac{\log_{10} x}{x}\, dx$

Solution. We rewrite with the formula $\log_b(x) = \frac{\ln(x)}{\ln(b)}$, and u-substitute $u = \ln(x)$.

$$\begin{aligned}
\int \frac{\log_{10} x}{x}\, dx &= \int \frac{1}{\ln(10)} \cdot \frac{\ln(x)}{x}\, dx \\
&= \int \frac{1}{\ln(10)} \cdot \frac{\ln(x)}{x}\, dx \\
&= \int \frac{1}{\ln(10)} u\, du \\
&= \frac{1}{2\ln(10)} u^2 + C \\
&= \frac{1}{2\ln(10)} \ln(x)^2 + C.
\end{aligned}$$

7. Solve the inequality $1 < e^{4x-2} < 2$, for x.

Solution. Since $\ln(\cdot)$ is always increasing, we may take the logarithm of both sides, which gives $\ln(1) = 0 < 4x - 2 < \ln(2)$. So:

$$\frac{1}{2} < x < \frac{\ln(2) + 2}{4}.$$

8. Solve the following equations:

(a) $e^{4x-6} = 8$.

Solution. Taking $\ln(\cdot)$ of both sides: $4x - 6 = \ln(8)$, so $x = \frac{\ln(8)+6}{4}$.

(b) $e - e^{-4x} = 4$.

Solution. Isolate the exponential. First: $e^{-4x} = e - 4$. Then $-4x = \ln(e - 4)$. So $x = \frac{-1}{4} \ln(e - 4)$.

(c) $\ln(x) + \ln(x - 1) = 1$.

Solution. Combine logarithms. First $\ln(x(x - 1)) = 1$. So $x(x - 1) = e^1 = e$. So $x^2 - x - e = 0$. The quadratic formula says the solutions are $x = \frac{1+\sqrt{1+4e}}{2}$ and $x = \frac{1-\sqrt{1+4e}}{2}$. We throw out the second solution, since it is a negative number and therefore not in the domain of the $\ln(x)$ appearing in the original equation. The only solution is $x = \frac{1+\sqrt{1+4e}}{2}$.

9. Differentiate the following functions:

(a) $G(x) = 4^{C/x}$, where C is a constant

Solution. I will rewrite $G(x) = (e^{\ln(4)})^{C/x} = e^{\frac{C\ln(4)}{x}}$. Then

$$G'(x) = \frac{-C\ln(4)}{x^2} e^{\frac{C\ln(4)}{x}}.$$

(b) $y = x^x$

Solution. I will rewrite $y = (e^{\ln(x)})^x = e^{x\ln(x)}$. Then $y' = (\ln(x) + 1)e^{x\ln(x)}$.

(c) $y = (\sin x)^{\ln x}$

Solution. I will rewrite $y = (e^{\sin(x)})^{\ln(x)} = e^{\sin(x)\ln(x)}$. Then $y' = (\cos(x) + \sin(x)\frac{1}{x})e^{\sin(x)\ln(x)}$.

(d) $y = (3x^2 + 5)^{\frac{1}{x}}$

Solution. I will write $y = (e^{\ln(3x^2+5)})^{1/x} = e^{\frac{1}{x}\ln(3x^2+5)}$. Therefore

$$y' = \left(\frac{-1}{x^2}\ln(3x^2+5) + \frac{1}{x} \cdot \frac{1}{3x^2+5} \cdot (6x)\right) e^{\frac{1}{x}\ln(3x^2+5)}.$$

10. Find y' if $x^y = y^x$.

Solution. Let's rewrite as $e^{\ln(x)y} = e^{\ln(y)x}$. Then $\ln(x)y = \ln(y)x$, by taking logarithm of both sides. This now looks like a more typical implicit differentiation problem.

$$\begin{aligned} \ln(x)y &= \ln(y)x \\ \Rightarrow \frac{1}{x}y + \ln(x)y' &= \frac{y'}{y}x + \ln(y) \\ \Rightarrow y' &= \frac{y/x - \ln(y)}{x/y - \ln(x)}. \end{aligned}$$

11. A computer is programmed to inscribe a series of rectangles in the first quadrant under the curve of $y = e^{-x}$. What is the area of the largest rectangle that can be inscribed?

Solution. It is best to choose $(0, 0)$ as one of the corners of the rectangle. Since $y = e^{-x}$ is always decreasing on the x-axis, this will allow for the most possible area. Let $(x, 0)$ be the other unknown corner of the base of the rectangle, $x \geq 0$. Then the rectangle has height e^{-x}. (Given corners $(0, 0)$ and $(x, 0)$, this is the highest we can make the rectangle to remain under the curve.)

So: the area under the triangle is $A(x) = xe^{-x}$. We compute $A'(x) = e^{-x} - xe^{-x}$. Then $A'(x) = 0$ when $1 - x = 0$, i.e. when $x = 1$. And from factoring $A'(x) = e^{-x}(1 - x)$, we see that $A'(x) > 0$ on $(0, 1)$ and $A'(x) < 0$ on $(1, +\infty)$. Therefore $x = 1$ is an absolute max of $A(x)$ on $(0, +\infty)$. The most possible area is $A(1) = 1/e$.

12. Let $a \neq -1$ be a constant. Calculate $\int \frac{x}{a} + \frac{a}{x} + x^a + a^x + ax\,dx$.

Solution.

$$\begin{aligned}\int \frac{x}{a} + \frac{a}{x} + x^a + a^x + ax\,dx &= \int \frac{x}{a} + \frac{a}{x} + x^a + e^{\ln(a)x} + ax\,dx \\ &= \frac{x^2}{2a} + a\ln|x| + \frac{1}{a+1}x^{a+1} + \frac{1}{\ln(a)}e^{\ln(a)x} + \frac{a}{2}x^2 + C.\end{aligned}$$

13. Sketch the graph of $f(x) = \ln(1+x^2)$ using the curve sketching techniques you learned in Chapter 3.

Solution. First, we see that $f(x)$ has domain $(-\infty, +\infty)$, since the input $1+x^2$ into $\ln(x)$ is always positive. Furthermore, $f(x) \geq 0$ always, since $1+x^2 \geq 0$, so $f(x) = \ln(1+x^2) \geq \ln(1) = 0$. And $f(x) = 0$ only when $1+x^2 = 1$, which only happens when $x = 0$.

We compute $f'(x) = \frac{2x}{1+x^2}$. Since this denominator $1+x^2$ is always positive, then by looking at the numerator we can see $f'(x) > 0$ on $(0, +\infty)$ and $f'(x) < 0$ on $(-\infty, 0)$.

Solution. Now we compute

$$f''(x) = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}.$$

Again, the denominator is always positive. We see where $f''(x)$ is positive or negative by analyzing the numerator. $f''(x) > 0$ when $1 - x^2 > 0$, which happens when $x^2 < 1$, which happens when $-1 < x < 1$. And so $f''(x) < 0$ on $(-\infty, -1) \cup (1, +\infty)$.

$f(x)$ diverges to $+\infty$ as x approaches $\pm\infty$.

Graph yourself: `https://www.desmos.com/calculator`.

Inverse trig functions.

1. What is the domain and range of $f(x) = \arcsin(x)$? What is the domain and range of $g(x) = \arctan(x)$?

   Solution. $f(x)$ has domain $[-1, 1]$ and range $[\frac{-\pi}{2}, \frac{\pi}{2}]$. And $g(x)$ has domain $(-\infty, +\infty)$ and range $(\frac{-\pi}{2}, \frac{\pi}{2})$.

2. What is $\arcsin(\frac{1}{2})$?

   Solution. $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$, because that is the angle in the range $[\frac{-\pi}{2}, \frac{\pi}{2}]$ which has sine equal to $\frac{1}{2}$.

3. Compute $\tan(\arcsin(\frac{4}{5}))$ explicitly. What about $\tan(\arcsin(x))$?

   Solution. Drawing a triangle, we find $\tan(\arcsin(\frac{4}{5})) = \frac{4}{3}$. In general, $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$.

4. Figure out $\frac{d}{dx}\arctan(x)$ by the following steps. Let $y = \arctan(x)$. Then $\tan(y) = x$. Use implicit differentiation on the equation $\tan(y) = x$ to find $\frac{dy}{dx}$, and then convert your formula for $\frac{dy}{dx}$ to be completely in terms of the variable x.

   Solution. We have: $\sec^2(y) \cdot y' = 1$, so $\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \cos^2(y)$. Converting, we have

$$\frac{dy}{dx} = \cos(\arctan(x))^2 = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}.$$

5. Compute: $\lim_{x\to+\infty} e^{-x}\arctan(x)$.

   Solution. Squeeze Theorem. we have $\frac{-\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$, and so $\frac{-\pi}{2}e^{-x} \leq e^{-x}\arctan(x) \leq \frac{\pi}{2}e^{-x}$. Both of the blue terms have limit 0 as $x \to +\infty$, so $\lim_{x\to+\infty} e^{-x}\arctan(x) = 0$.