1. Determine the derivatives of the following functions:

   (a) $f(x) = \sin^{-1}(4x^2)$

   Solution.

   $$f'(x) = \frac{1}{\sqrt{1-16x^4}} \cdot (8x).$$

   (b) $g(s) = \cos^{-1}(s)\ln(2s)$.

   Solution.

   $$g'(s) = \frac{-1}{\sqrt{1-s^2}} \cdot \ln(2x) + \cos^{-1}(s) \cdot \frac{1}{s}.$$

   Notice that $\ln(2s) = \ln(2) + \ln(s)$, so $\frac{d}{ds}\ln(2s) = \frac{1}{s}$.

   (c) $y = (\tan^{-1} x)^2$

   Solution.

   $$y' = 2\arctan(x) \cdot \frac{1}{1+x^2}.$$

   (d) $f(x) = \arcsin(e^x)$

   Solution.

   $$f'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x.$$

   (e) $y = \arctan\sqrt{\frac{1-x}{1+x}}$

   Solution. Chain rule.

   $$y' = \frac{1}{1+\frac{1-x}{1+x}} \cdot \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \cdot \frac{-(1+x)-(1-x)}{(1+x)^2}.$$

   (f) $y = \tan^{-1}\left(\frac{x}{a}\right) + \ln\sqrt{\frac{x-a}{x+a}}$

   Solution. Rewrite:

   $$y = \arctan(x/a) + \frac{1}{2}\ln\left(\frac{x-a}{x+a}\right) = \arctan(x/a) + \frac{1}{2}\ln(x-a) - \frac{1}{2}\ln(x+a).$$

   Now it is clearer to compute:

   $$y' = \frac{1/a}{1+(x/a)^2} + \frac{1}{2(x-a)} - \frac{1}{2(x+a)}.$$

2. Find the absolute max and absolute min of the function $f(x) = e^x - ex$ on the interval $0 \leq x \leq 5$.

   Solution. Endpoints: $f(0) = 1$ and $f(5) = e^5 - 5e \cong 134$. Critical points: $f'(x) = e^x - e = 0$ when $x = 1$. And $f(1) = e - e = 0$. The absolute max is at $x = 5$ and the absolute min is at $x = 1$.

3. Find y' if $\tan^{-1}(x^2y) = 2x + xy$.

Partial solution. Implicit diff.

$$\frac{1}{1+(x^2y)^2}\left(2xy + x^2y'\right) = 2 + y + xy'.$$

Now solve for y'.

4. Find and equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

Solution. We have:

$$y'(x) = \frac{-3}{\sqrt{1-(x/2)^2}} \cdot (1/2).$$

Therefore, the tangent slope is

$$y'(1) = \frac{-3}{\sqrt{3}/2} \cdot \frac{1}{2} = -\sqrt{3}.$$

So the tangent line is:

$$y = \pi - \sqrt{3}(x-1).$$

5. Show that there is exactly one root of the equation $\ln(x) = 3 - x$ and that it lies between 1 and e.

Solution. Intermediate Value Theorem. Let $f(x) = \ln(x) - (3 - x)$. We need to show that $f(x) = 0$ only once, somewhere between 1 and e. We compute $f(1) = -2$ and $f(e) = -2 + e > 0$. So the Intermediate Value Theorem says there is a root somewhere in $(1, e)$. And: $f'(x) = \frac{1}{x} + x$. We see that $f'(x) > 0$ always on $(0, +\infty)$, which is the entire domain of $f(x)$. Since $f(x)$ is always increasing on its domain, it can only touch the x-axis once. There $f(x)$ has exactly one root on its domain, and the root is in the interval $(1, e)$.

6. Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{(y-1)^2+1}\,dy$

Solution. We can u-substitute, or recognize this as a linear substitution into the function $\frac{1}{1+x^2}$.

$$\int \frac{1}{(y-1)^2+1}\,dy = \arctan(y-1) + C.$$

(b) $\displaystyle\int_0^{\sqrt{3}/4} \frac{1}{1+16x^2}\,dx$

Solution. Rewrite:

$$\int_0^{\sqrt{3}/4} \frac{1}{1+16x^2}\,dx = \int_0^{\sqrt{3}/4} \frac{1}{1+(4x)^2}\,dx.$$

We can either use u-substitution, or recognize that the antiderivative will be $\arctan(4x)$ scaled by a suitable constant. If we multiply by a factor of $\frac{1}{4}$, the derivative will match $\frac{1}{1+(4x)^2}$; the function $\frac{1}{4}\arctan(4x)$ is an antiderivative of $\frac{1}{1+(4x)^2}$. Therefore:

$$\int_0^{\sqrt{3}/4} \frac{1}{1+16x^2}\,dx = \left(\frac{1}{4}\arctan(4x)\right)\Big|_{x=0}^{x=\sqrt{3}/4} = \frac{1}{4}(\arctan(\sqrt{3}) - \arctan(0)) = \frac{1}{4}\cdot\frac{\pi}{3} = \frac{\pi}{12}.$$

(c) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

Partial solution. Split up the fraction.

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx.$$

The first integral is simply $\arctan(x)$. For the second, u-substitute $u = 1+x^2$. We will end up finding:

$$\int \frac{1+x}{1+x^2}\,dx = \arctan(x) + \frac{1}{2}\ln(1+x^2) + C.$$

(d) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

Partial solution. u-substitute $u = \cos(x)$, which gives $-du = \sin(x)$, and new endpoints $u = 1$ to $u = 0$. So:

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_1^0 \frac{-1}{1+u^2}\,du = \frac{\pi}{4}.$$

(e) $\displaystyle\int \frac{1}{\sqrt{1-x^2}\sin^{-1} x}\,dx$

Partial solution. Correct choice of u-substitution is $u = \sin^{-1}(x)$. The answer is $\ln(\sin^{-1}(x)) + C$.

(f) $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

No solution. Answer is approximately 4.1888.

(g) $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$

Solution. Good technique to remember. The idea is to see the hidden square.

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \int \frac{e^{2x}}{\sqrt{1-(e^{2x})^2}}\,dx.$$

Now we can u-substitute $u = e^{2x}$, which gives $du = 2e^{2x}\,dx$, so $\frac{1}{2}du = e^{2x}\,dx$. Therefore:

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \int \frac{1}{2\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(e^{2x}) + C.$$

(h) $\displaystyle\int \frac{x}{1+x^4}\,dx$

Solution. Good technique to remember. The idea is to see the hidden x^2 term.

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx.$$

Now we see that we should substitute $u = x^2$, which gives $\frac{1}{2}du = x\,dx$. So:

$$\int \frac{x}{1+x^4}\,dx = \int \frac{x}{1+(x^2)^2}\,dx$$

(i) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ for $a > 0$

Solution. a is just a constant here. Substitute $a = 3$ for example, and this problem might seem a little more concrete.

We rewrite, so the integrand matches better the antiderivative of $\arcsin(x)$. We have:

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-(x/a)^2}} = \frac{1}{|a|\sqrt{1-(x/a)^2}} = \frac{1}{a\sqrt{1-(x/a)^2}}.$$

We used the equality $|a| = a$, since $a > 0$. Therefore:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-(x/a)^2}}\,dx.$$

Now we u-substitute $u = x/a$, which gives $du = (1/a)\,dx$. Therefore:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du$$
$$= \arcsin(u) + C = \arcsin(x/a) + C.$$

(j) $\displaystyle\int \frac{\sin(\arctan(x))}{2+2x^2}\,dx$

Solution. We factor out a 2 and u-subsitute $u = \arctan(x)$.

$$\begin{aligned}\int \frac{\sin(\arctan(x))}{2+2x^2}\,dx &= \int \frac{\sin(\arctan(x))}{2(1+x^2)}\,dx \\ &= \int \frac{1}{2}\sin(u)\,du \\ &= \frac{-1}{2}\cos(u) + C \\ &= -\frac{1}{2}\cos(\arctan(x)) + C.\end{aligned}$$

7. Find $\frac{dq}{dp}$ if $\arcsin(pq) + q^2 = \frac{q}{p}$.

Solution. Taking d/dp:

$$\frac{q + pq'}{\sqrt{1-(pq)^2}} + 2q = \frac{q'}{p} + \frac{-q}{p^2}.$$

Solving for q' should give:

$$q' = \frac{\frac{-q}{p^2} + \frac{-q}{\sqrt{1-(pq)^2}} - 2q}{\frac{p}{\sqrt{1-(pq)^2}} - \frac{1}{p}}.$$

8. Eliminate the trig functions from the following expressions:

(a) $\tan(\sin^{-1} x)$

Solution. $\tan(\sin^{-1} x) = \frac{x}{\sqrt{1-x^2}}$, by drawing a triangle.

(b) $\sin(\tan^{-1} x)$

Solution. $\sin(\tan^{-1} x) = \frac{x}{\sqrt{x^2+1}}$, by a drawing a triangle.

(c) $\sin(2 \arccos x)$

Solution. There is really only one way to do this correctly.

$$\sin(2 \arccos x) = 2 \sin(\arccos(x)) \cos(\arccos(x)) = 2 \cdot \sqrt{1-x^2} \cdot x.$$

9. If $g(x) = x \sin^{-1}(x/4) + \sqrt{16-x^2}$, find the equation of the line tangent to $g(x)$ at $x = 2$.

Partial solution. We have

$$g'(x) = \frac{1}{2}x^2 \arcsin(x/4) + 4 \arcsin(x/4) + \frac{3}{4}x\frac{1}{\sqrt{16-x^2}}.$$

So $g'(2) = \pi + 3^{3/2}$. And we have $g(2) = \frac{\pi}{3} + 2\sqrt{3}$. Therefore the tangent line is

$$y = (\pi + 3^{3/2})(x-2) + \frac{\pi}{3} + 2\sqrt{3}.$$

10. Sketch the function $f(x) = \tan^{-1}(x) - x$ using the techniques you learned in Chapter 3.

Solution. We compute $f'(x) = \frac{1}{1+x^2} - 1$. Notice that $0 < \frac{1}{1+x^2} \leq 1$; since $x^2 \geq 0$, the denominator is always bigger than or equal to 1. Therefore $f'(x) \leq 0$ always, and $f'(x) = 0$ only when $x = 0$. So the function $f(x)$ is always decreasing.

We also compute $f''(x) = \frac{-2x}{(1+x^2)^2}$. The denominator of $f''(x)$ is always positive. By looking at the numerator, we see that $f''(x) < 0$ on $(0, +\infty)$ and $f''(x) > 0$ on $(-\infty, 0)$.

Plugging in $x = 0$ gives $f(0) = 0$. Since $f(x)$ is always decreasing, this is the only place that $f(x)$ can cross the x-axis. Therefore $f(x)$ is positive on the negative x-axis, and is negative on the positive x-axis.

Graph yourself: `https://www.desmos.com/calculator`.