Volumes.

Key point about these volume/revolution questions. They are solved pretty much the exact same way as the prior questions about finding the area between two curves. If we have a top curve $f(x)$ and a bottom curve $g(x)$ on an x-range $a \leq x \leq b$, then the area between the two curves is

$$A = \int_a^b f(x) - g(x)\, dx.$$

And, the *volume* of the region given by spinning this area around the *x-axis* is:

$$V = \int_a^b \pi f(x)^2 - \pi g(x)^2\, dx,$$

which just comes from the formula for the area of a circle. In other words, it's pretty much the same setup, we are just replacing $f(x)$ by $\pi f(x)^2$!

1. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the x-axis.

   Solution. This is a sphere with radius 3. The volume is $V = \frac{4}{3}\pi(3)^3 = 36\pi$.

2. Find the volume of the solid obtained by revolving the region enclosed by $x = \sqrt{2\sin(2y)}$, $0 \leq y \leq \pi/2$, and $x = 0$ about the y-axis.

   Setup.
   $$V = \int_0^{\pi/2} \pi\left(\sqrt{2\sin(2y)}\right)^2\, dy = 2\pi \int_0^{\pi/2} \sin^2(2y)\, dy = 2\pi \int_0^{\pi/2} \frac{1 - \cos(4y)}{2}\, dy.$$

3. Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{\cos(x)}$, $0 \leq x \leq \pi/2$, and $y = 0$ about the x-axis.

   Setup.
   $$V = \int_0^{\pi/2} \pi\left(\sqrt{\cos x}\right)^2\, dx = \int_0^{\pi/2} \cos(x)\, dx.$$

4. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by $y = 4-x^2$ and $y = 2 - x$ about the x-axis.

Setup. Solving the equation $4 - x^2 = 2 - x$, we see that the graphs of the two functions intersect when $x = -1$ and $x = 2$. And in the interval $(-1, 2)$, the graph of $y = 4 - x^2$ lies above $y = 2 - x$, which we can see by plugging in test points in to $y = 4 - x^2$ and $y = 2 - x$. Therefore the volume is:

$$V = \int_{-1}^{2} \left(\pi(4 - x^2)^2 - \pi(2 - x)^2\right) \, dx.$$

5. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by $y = x^2$ and the line $y = 1$ about the line $y = -2$.

Setup. The x-range is the interval $[-1, 1]$, from solving the equation $x^2 = 1$.

$$V = \int_{-1}^{1} \left(\pi(1 - (-2))^2 - \pi(x^2 - (-2))^2\right) \, dx = \int_{-1}^{1} \left(9\pi - \pi(x^2 + 2)^2\right) \, dx.$$

6. Write down an integral that represents the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about the line $x = 4$.

Setup.

$$V = \int_{0}^{2} \left(\pi 4^2 - \pi(y^2)^2\right) \, dy.$$

Final Exam Review.

7. A rectangle is to be inscribed under the arch of the curve $y = 1 - x^2$ from $x = -1$ to $x = 1$. What are the dimensions of the rectangle with largest area, and what is the largest area?

Solution. Draw a careful picture (See Oct. 28 worksheet # 1 and # 2 for a similar setup.) If $(x, 0)$ is the lower right corner of the unknown rectangle to maximize, the area of the rectangle is

$$A(x) = 2x(1 - x^2) = 2x - 2x^3.$$

We need to maximize $A(x)$ over $0 \leq x \leq 1$.

$$A'(x) = 2 - 6x^2 = 0,$$

which gives $x = 3^{-1/2}$ as the solution. The maximum area is $A(3^{-1/2}) \cong 0.77$.

8. Compute the first derivative of the following functions

   (a) $f(x) = \ln(x^3 - 4x)\sin(2x)$

   Solution.
$$f'(x) = \frac{(3x^2 - 4)\sin(2x)}{x^3 - 4x} + 2\ln\left(x^3 - 4x\right)\cos(2x).$$

   (b) $g(s) = \sin(\cos(e^{\sin(s)}))$

   Solution.
$$g'(s) = -\cos\left(\cos\left(e^{\sin(s)}\right)\right)\sin\left(e^{\sin(s)}\right)e^{\sin(s)}\cos(s).$$

   (c) $h(t) = \sqrt{\frac{t-1}{t^2+2}}$

   Solution.
$$h'(t) = \frac{\sqrt{\frac{t-1}{t^2+2}}\left(\frac{t^2}{2} - t(t-1) + 1\right)}{(t-1)(t^2+2)}.$$

   I used this program to compute this: `https://www.sympy.org/en/index.html`. It may or may not be natural to get the algebra in your answer to look like this.

   (d) $g(t) = \frac{\cos(2t)}{t-5}$

   Solution.
$$g'(t) = \frac{(10 - 2t)\sin(2t) - \cos(2t)}{(t-5)^2}.$$

   (e) $F(x) = \displaystyle\int_3^{x^3} e^{4t^2}\, dt.$

   Solution.
$$F'(x) = 3x^2 e^{4x^6}.$$

9. You are videotaping a race from the inside of a blimp 132 ft directly above the finish line, following a car that is moving at 180 mph (264 ft/s). How fast will your camera angle be changing when the car is finishing the race?

Solution. Draw a careful picture. Call θ the unknown angle, and x the distance of the car to the finish line. We have the equation $\tan(\theta) = \frac{x}{132}$, so $\theta = \arctan(\frac{x}{132})$. Therefore from the Chain Rule

$$\frac{d\theta}{dt} = \frac{1}{1 + (x/132)^2} \cdot \frac{1}{132} \cdot \frac{dx}{dt}.$$

At the finish line, $x = 0$, and $\frac{dx}{dt} = -264$. So

$$\frac{d\theta}{dt} = \frac{1}{132} \cdot (-264) = -2 \text{ (radians/sec.)}$$

10. Let $f(x) = 3x^2 + 4x$. Use the definition of the derivative to compute $f'(2)$.

No solution. See Sept. 23 worksheet for comparison. You know that $f'(x) = 6x + 4$, so the answer of your limit computation should be $f'(2) = 16$.

11. Compute the following limits.

(a) $\displaystyle\lim_{\theta \to 0} \frac{\tan(\theta)}{\theta + \sin(\theta)}$

Answer. Straightforward L'Hôpital. The answer is 1.

(b) $\displaystyle\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$.

Solution. Square root conjugate method. Then pull out leading terms. You'll want to be able to solve a problem like this yourself!

$$\begin{aligned}
\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} &= \lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \cdot \left(\frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \right) \\
&= \lim_{x \to \infty} \frac{(x^2 + x - 1) - (x^2 - x)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\
&= \lim_{x \to \infty} \frac{2x - 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\
&= \lim_{x \to \infty} \frac{x(2 - x^{-1})}{\sqrt{x^2(1 + x^{-1} + x^{-2})} + \sqrt{x^2(1 - x^{-1})}} \\
&= \lim_{x \to \infty} \frac{x(2 - x^{-1})}{|x|\sqrt{1 + x^{-1} + x^{-2}} + |x|\sqrt{1 - x^{-1}})} \\
&= \lim_{x \to \infty} \frac{x(2 - x^{-1})}{x\sqrt{1 + x^{-1} + x^{-2}} + x\sqrt{1 - x^{-1}})} \\
&= \lim_{x \to \infty} \frac{(2 - x^{-1})}{\sqrt{1 + x^{-1} + x^{-2}} + \sqrt{1 - x^{-1}})} \\
&= \frac{2}{1 + 1} \\
&= 1.
\end{aligned}$$

(c) $\displaystyle\lim_{u \to 0} \frac{5 - 5\cos(u)}{e^u - u - 1}$.

Solution. L'Hôpital ($2\times$).

$$\lim_{u \to 0} \frac{5 - 5\cos(u)}{e^u - u - 1} = \lim_{u \to 0} \frac{5\sin(u)}{e^u - 1} = \lim_{u \to 0} \frac{5\cos(u)}{e^u} = 5.$$

(d) $\lim_{x\to\infty} \frac{e^x+1}{e^x-1}$

Solution. Pull out leading terms.

$$\begin{aligned}\lim_{x\to\infty} \frac{e^x+1}{e^x-1} &= \lim_{x\to\infty} \frac{e^x(1+e^{-x})}{e^x(1-e^{-x})} \\ &= \lim_{x\to\infty} \frac{1+e^{-x}}{1-e^{-x}} \\ &= \frac{1+0}{1-0} \\ &= 1.\end{aligned}$$

12. Compute the following integrals

(a) $\displaystyle\int e^x \sin(e^x)\, dx.$

Answer. $-\cos(e^x) + C$, by u-substituting $u = e^x$.

(b) $\displaystyle\int_{-1}^{1} \frac{dx}{3x-4}.$

Answer. $\frac{-\ln(7)}{3}$, by u-substituting $u = 3x - 4$.

(c) $\displaystyle\int_{1}^{3} \frac{s^2 + 2\sqrt{s} - s + 3}{4s}.$

Answer. First simplify to $\int_1^3 \frac{1}{4}s + \frac{1}{2}s^{-1/2} - \frac{1}{4} + \frac{3}{4}s^{-1}\, ds$. Now it is straightforward to compute

$$\int_1^3 \frac{s^2 + 2\sqrt{s} - s + 3}{4s} = -\sqrt{3} + \frac{3}{4}\ln(3) + \frac{3}{2}.$$

13. Sketch the graph of the function $f(x) = \frac{x^2-4}{2x}$.

Partial solution. The function is equal to 0 at $x = 2$ and $x = -2$, and is not defined at $x = 0$. Plugging in test points, we find that $f(x) > 0$ on $(-2, 0) \cup (2, +\infty)$ and $f(x) < 0$ on $(-\infty, -2) \cup (0, 2)$.

There is a vertical asymptote at $x = 0$. Simplifying, $f(x) = \frac{1}{2}x - \frac{2}{x}$, we see $f(x)$ has a slant asymptote of $y = \frac{1}{2}x$ as $x \to \pm\infty$.

Using our simplified formula, $f'(x) = \frac{1}{2} + \frac{2}{x^2}$. Therefore $f(x)$ is always increasing (since $2/x^2$ is always positive). And $f''(x) = \frac{-4}{x^3}$. Therefore $f(x)$ is concave down on $(0, +\infty)$ and is concave up on $(-\infty, 0)$.

Graph: `https://www.desmos.com/calculator`.