1. Use any method to write down integrals that represent the volume of the following solids.

   (a) The solid obtained by rotating the region bounded by the x and y axes and the graph of $y = 3 - 3x$ about the y-axis.

   Solution. Solving for x in terms of y gives $x = 1 - \frac{1}{3}y$. The y-range is $0 \le y \le 3$, which we find by setting $3 - 3x = 0$ and plugging $x = 0$ into $y = 3 - 3x$. Therefore, the volume is:

   $$V = \int_0^3 \pi(1 - \frac{1}{3}y)^2 \, dy.$$

   (b) Let T be the triangle enclosed by $1 \le x \le 2$ and $0 \le y \le 3x - 3$.

   i. The solid obtained by rotating T around the x-axis.
   ii. The solid obtained by rotating T around the y-axis.
   iii. The solid obtained by rotating T around the line $x = -1$.
   iv. The solid obtained by rotating T around the line $y = -2$.

   Solution.
   i. $V = \int_1^2 \pi(3 - 3x)^2 \, dx$.
   ii. Solving the line equation $y = 3 - 3x$ for x gives: $x = 1 - \frac{1}{3}y$. By plugging in $x = 1$ and $x = 2$ to the equation $y = 3 - 3x$, the y-range is $-3 \le y \le 0$. The volume integral is $V = \int_{-3}^0 \pi(1 - \frac{1}{3}y)^2 \, dy$.
   iii. $V = \int_{-3}^0 \pi(1 - \frac{1}{3}y - (-1))^2 \, dy = \int_{-3}^0 \pi(2 - \frac{1}{3}y)^2 \, dy$.
   iv. $V = \int_1^2 \pi(3 - 3x - (-2))^2 \, dx = \int_1^2 \pi(5 - 3x)^2 \, dx$.

   (c) The solid obtained by rotating the region enclosed by $y = x$ and $y = \sqrt{x}$ about the line $x = 5$.

   Solution. Solving for x in terms of y, the function $y = x$ becomes $x = y$, and the function $y = \sqrt{x}$ becomes $x = y^2$. From the perspective of the line $x = 5$, the outer radius is $x = y^2$. To find the points of intersection, we set $y = y^2$, and find $y = 0$ and $y = 1$, so the y-range is $0 \le y \le 1$. The volume is:

   $$V = \int_0^1 \pi(5 - y^2)^2 - \pi(5 - y)^2 \, dy.$$

   (d) The solid obtained by rotating the region enclosed by $y = -(x^2 - 2x)$ and the x-axis about the line $x = 3$.

   Solution. To solve for x in terms of y, we complete the square. We have:

   $$y = -((x - 1)^2 - 1) \Rightarrow y = 1 \pm \sqrt{1 - y}.$$

   There are two x-curves: $x = 1 + \sqrt{1 - y}$ and $x = 1 - \sqrt{1 - y}$. From the perspective of the line $x = 3$, the top curve is $x = 1 - \sqrt{1 - y}$. Looking at the graph, the upper y-bound is $y = 1$ and the lower bound is $y = 0$. The volume is:

   $$V = \int_0^1 \pi(1 - \sqrt{1 - y})^2 - \pi(1 + \sqrt{1 - y})^2 \, dy.$$

   (e) The solid obtained by rotating the region enclosed by $x = 2 - y^2$, $x = y^4$; about the y-axis.

   Solution. The curves intersect at $y = 1$ and $y = -1$. The outer radius is $x = 2 - y^2$. The volume is

   $$V = \int_{-1}^1 \pi(2 - y^2)^2 - \pi(y^4)^2 \, dy.$$

2. Compute the average value of the following functions on the given interval.

   (a) The function $f(x) = \sin(2x)$ on the interval $[0, \pi/2]$.

   Solution.

   $$\frac{1}{\pi/2 - 0} \int_0^{\pi/2} \sin(2x)\, dx = \frac{2}{\pi}.$$

   (b) The function $f(x) = x^2 + 3$ on the interval $[-1, 1]$.

   Solution.

   $$\frac{1}{1-(-1)} \int_{-1}^{1} x^2 + 3\, dx = \frac{10}{3}.$$

Final Exam Review

3. Consider the curve $2yx + 3x^2y = \sin(xy)$. Find $\frac{dy}{dx}$.

Solution. We have $2y'x + 2y + 6xy + 3x^2y' = \cos(xy)(y + xy')$. Therefore

$$y' = \frac{y\cos(xy) - 6xy}{2x + 3x^2 - x\cos(xy)}.$$

4. Find the absolute max and the absolute min of the function $f(x) = x^3 - 2x$ on the interval $[0, 4]$.

Solution. We compute $f(0) = 0$, and $f(4) = 56$. And $f'(x) = 3x^2 - 2 = 0$ when $x = \sqrt{2/3}$. And $f(\sqrt{2/3}) \cong -1.09$. So the max is at $x = 4$ and the min is at $x = \sqrt{2/3}$.

5. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s, how fast is the water level rising when the water is 5 cm deep? (Recall that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

Solution. The volume of a cone is $V = \frac{1}{3}\pi r^2$ (we know how to derive this ourselves now!) We have $h/r = 10/3$ at all times by similar triangles, so $r = \frac{3}{10}h$. So:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{10}h\right)^2 h = \frac{3\pi}{100}h^3.$$

So:

$$\frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt},$$

so at this particular moment in time:

$$2 = \frac{9\pi}{100}\cdot 5^2 \cdot \frac{dh}{dt},$$

and therefore

$$\frac{dh}{dt} = \frac{8}{9\pi}\ \text{cm/s}.$$

6. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2 respectively.

Solution. We have: $a + b + c = 4$. And $y' = 2ax + b$. And $6 = -2a + b$. And $-2 = 10a + b$. Solving this system of equations gives $a = \frac{-2}{3}$, $b = \frac{14}{3}$, $c = 0$. The parabola is

$$y = \frac{-2}{3}x^2 + \frac{14}{3}x.$$

7. Compute the following limits, if they exist. If the limit does not exist, decide whether it is ∞, $-\infty$ or neither.

(a) $\displaystyle\lim_{v\to 4^+} \frac{4-v}{|4-v|}$.

Solution. On the interval $[4, +\infty)$, we have $4 - v \leq 0$, so $|4 - v| = -(4 - v)$. Therefore:

$$\lim_{v\to 4^+} \frac{4-v}{|4-v|} = \lim_{v\to 4^+} \frac{4-v}{-(4-v)} = \lim_{v\to 4^+} -1 = -1.$$

(b) $\displaystyle\lim_{x\to 0} \cos\left(\frac{2}{x}\right) x^4$.

Solution. Squeeze Theorem.

$$-x^4 \leq \cos\left(\frac{2}{x}\right) x^4 \leq x^4.$$

And $\lim_{x\to 0} x^4 = \lim_{x\to 0} -x^4 = 0$. Therefore $\lim_{x\to 0} \cos\left(\frac{2}{x}\right) x^4 = 0$.

(c) $\displaystyle\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3}$.

Solution. As $x \to 1^+$, the numerator $x^2 - 9$ approaches $(1)^2 - 9 = -8$. And the denominator $(x^2 + 2x - 3) = (x + 3)(x - 1)$ is positive to the right of $x = 1$. Therefore $\lim_{x\to 1^+} \dfrac{x^2-9}{x^2+2x-3} = -\infty$.

8. Does the function $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}$ have any discontinuities? If so, determine whether the discontinuity is a removable discontinuity, a jump discontinuity or an infinite discontinuity.

Solution.

$$f(x) = \frac{x(x^2 - x - 2)}{x - 2} = \frac{x(x-2)(x+1)}{x-2} = x(x+1), \text{ for } x \neq 2.$$

There is a removable discontinuity at $x = 2$.

9. Show that the equation $3x + 2\cos(x) + 5 = 0$ has exactly one real root.

Solution. Let $f(x) = 3x + 2\cos(x) + 5$. The function $f(x)$ has a root by the Intermediate Value Theorem, since $f(-90210)$ is negative and $f(90210)$ is positive. And $f'(x) = 3 - 2\sin(x) \geq 1$. Therefore the function is always increasing, so it can hit the x-axis at most once.

10. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$ and $2 \leq f'(x) \leq 5$ for all x in $(0, 4)$. Show that $9 \leq f(4) \leq 21$.

Solution. I'm sure you recognize this one from your midterm. The Mean Value Theorem says

$$\frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4} = f'(x)$$

for some $0 \leq x \leq 4$. Therefore

$$2 \leq \frac{f(4) - 1}{4} \leq 5,$$

so $8 \leq f(4) - 1 \leq 20$, so $9 \leq f(4) \leq 21$.

11. Find the point on the ellipse $\frac{x^2}{9} + y^2 = 1$ that is closest to the point $(2, 0)$.

Solution. We want to minimize the distance

$$d = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x-2)^2 + y^2} = d = \sqrt{(x-2)^2 + 1 - \frac{1}{9}x^2}.$$

Here, we used the equation $\frac{x^2}{9} + y^2 = 1$. To find the critical points, we set

$$d' = \frac{2(x-2) - \frac{2}{9}x}{2\sqrt{(x-2)^2 + 1 - \frac{1}{9}x^2}} = 0.$$

After cross-multiplying, we can solve $x = \frac{9}{4}$. We want to maximize over the interval $-3 \leq x \leq 3$; this is the possible range of x-values. We compute $d(3) = 1$ and $d(-3) = 5$. And $d(\frac{9}{4}) = \frac{1}{\sqrt{2}}$. Therefore the maximum distance is at the point $(x, y) = (3, 0)$.

We found the range of x-values by analyzing the graph of the ellipse $\left(\frac{x}{3}\right)^2 + y^2 = 1$. This is the unit circle $x^2 + y^2 = 1$ stretched by 3 in the x-direction (see for example: $(\frac{3}{3}, 0) = (1, 0)$ is on the unit circle, and so $(3, 0)$ is on the ellipse.)

12. Find f if $f''(x) = 5x^3 + 6x^2 + 2$, with $f(0) = 3$ and $f(1) = -2$.

Solution. We have

$$f'(x) = \frac{5}{4}x^4 + 2x^3 + 2x + C_0$$

and

$$f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + C_0 x + C_1.$$

The equation $f(0) = 3$ gives $C_1 = 3$. And the equation $f(1) = -2$ gives $C_0 + \frac{19}{4} = -2$, so $C_0 = \frac{-27}{4}$. Therefore

$$f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3.$$

13. Find the area of the region bounded by the curves $y = e^x$, $y = e^{-x}$, $x = -2$ and $x = 1$.

Solution. The point of intersection is $x = 0$. On the interval $[-2, 0]$, the graph $y = e^{-x}$ is the top curve. On the interval $[0, 1]$, the graph $y = e^x$ is the top curve. The integral is:

$$A = \int_{-2}^{0} e^{-x}\, dx + \int_{0}^{1} e^x\, dx \cong 8.1.$$

14. Write an integral that represents the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the line $y = 2$.

Solution. The points of intersection are $x = 0$ and $x = 1$. Using test points, $y = x^2$ is the bottom curve. The volume is

$$V = \int_{0}^{2} \pi(x^2 - 2)^2 - \pi(\sqrt{x} - 2)^2\, dx.$$