1. Given that $\lim_{x \to -4} f(x) = 6$, $\lim_{x \to -4} g(x) = 0$, and $\lim_{x \to -4} h(x) = 1$, find the limits below. If the limit does not exist, explain why.

   (a) $\lim_{x \to -4} [f(x) - 4h(x)]$

   Solution. Using limits' additive and scaling properties,

$$\begin{aligned} \lim_{x \to -4} [f(x) - 4h(x)] &= \lim_{x \to -4} f(x) + \lim_{x \to -4} [-4h(x)] \\ &= \lim_{x \to -4} f(x) - 4 \lim_{x \to -4} h(x) \\ &= 6 - 4(1) \\ &= 2. \end{aligned}$$

   (b) $\lim_{x \to -4} [f(x)]^3$

   Solution. We can pull limits inside continuous functions. Since $g(t) = t^3$ is a continuous function, we have:

$$\lim_{x \to -4} [f(x)]^3 = \left(\lim_{x \to -4} f(x) \right)^3 = 6^3 = 216.$$

   (c) $\lim_{x \to -4} \dfrac{g(x)}{3h(x)}$.

   Solution. We are allowed to pull limits onto the numerator and denominator, since the limit of the denominator is not zero.

$$\lim_{x \to -4} \frac{g(x)}{3h(x)} = \frac{\lim_{x \to -4} g(x)}{\lim_{x \to -4} 3h(x)} = \frac{0}{3 \cdot 1} = 0.$$

   (d) $\lim_{x \to -4} \dfrac{h(x)}{2g(x)}$.

   Solution. The limit does not exist, since the denominator approaches zero and the numerator does not approach zero.

2. Evaluate each limit and justify each step by indicating the appropriate Limit Laws.

   (a) $\lim_{a \to 2} \dfrac{a^4 - 8a + 4}{3a^2 + 16}$

   Solution. We'll be very careful to do each maneuver with limits step by step.

$$\begin{aligned} \lim_{a \to 2} \frac{a^4 - 8a + 4}{3a^2 + 16} &= \frac{\lim_{a \to 2}(a^4 - 8a + 4)}{\lim_{a \to 2} 3a^2 + 16} \\ &= \frac{(\lim_{a \to 2} a)^4 - 8(\lim_{a \to 2} a) + 4}{3(\lim_{a \to 2} a)^2 + 16} \\ &= \frac{2^4 - 8 \cdot 2 + 4}{3 \cdot 2^2 + 16} \\ &\cong 0.143. \end{aligned}$$

(b) $\lim_{u\to -1} \sqrt{\frac{2u+5}{3u+11}}$.

Solution. Using limit laws:

$$\begin{aligned}
\lim_{u\to -1} \sqrt{\frac{2u+5}{3u+11}} &= \sqrt{\lim_{u\to -1} \frac{2u+5}{3u+11}} \\
&= \sqrt{\frac{\lim_{u\to -1}(2u+5)}{\lim_{u\to -1}(3u+11)}} \\
&= \sqrt{\frac{2(-1)+5}{3(-1)+11}} \\
&= \sqrt{\frac{3}{8}}.
\end{aligned}$$

3. Evaluate the following limit, if it exists. If the limit does not exist, explain why. If you use a theorem, clearly state which theorem you are using.

(a) $\lim_{x\to -6} \frac{\frac{1}{x}+\frac{1}{6}}{x+6}$

Solution. Let's combine the numerator into one fraction. Then the $(x+6)$ terms cancel. The only limit theorem we use here is that $f(x) = 1/6x$ is a continuous function at $x = -6$.

$$\begin{aligned}
\lim_{x\to -6} \frac{\frac{1}{x}+\frac{1}{6}}{x+6} &= \lim_{x\to -6} \frac{\frac{6+x}{6x}}{x+6} \\
&= \lim_{x\to -6} \frac{1}{6x} \\
&= \lim_{x\to -6} \frac{1}{6(-6)} \\
&= \frac{-1}{36}.
\end{aligned}$$

(b) $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution. Using **polynomial long division** (another algebra skill to brush up on), we find $\frac{x^3-1}{x-1} = x^2+x+1$. Therefore:

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} x^2+x+1 = 1^1+1+1 = 3.$$

(c) $\lim_{v\to \frac{1}{2}^-} \frac{|2v-1|}{2v-1}$.

Solution. Remember that

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

Therefore, plugging in $2v-1$ and solving the resulting inequality,

$$\begin{aligned}
|2v-1| &= \begin{cases} 2v-1, & 2v-1 \geq 0 \\ -(2v-1), & 2v-1 < 0 \end{cases} \\
&= \begin{cases} 2v-1, & v \geq \frac{1}{2} \\ -(2v-1), & v < \frac{1}{2} \end{cases}
\end{aligned}$$

Since we are taking the left-hand limit in the range $x < \frac{1}{2}$, we have

$$\lim_{v\to \frac{1}{2}^-} \frac{|2v-1|}{2v-1} = \lim_{v\to \frac{1}{2}^-} \frac{-(2v-1)}{2v-1} = \lim_{v\to \frac{1}{2}^-} -1 = -1.$$

(d) $\lim_{v\to\frac{1}{2}} \frac{|2v-1|}{2v-1}$.

Solution. Taking the right-hand limit, we similarly compute:

$$\lim_{v\to\frac{1}{2}^+} \frac{|2v-1|}{2v-1} = \lim_{v\to\frac{1}{2}^+} \frac{2v-1}{2v-1} = \lim_{v\to\frac{1}{2}^+} 1 = 1.$$

Therefore $\lim_{v\to\frac{1}{2}} \frac{|2v-1|}{2v-1}$ does not exist, because the corresponding right-hand and left-hand limits are not equal.

(e) $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right)$

Solution. Using the squeeze theorem:

$$\mathbf{-x^4} \le x^4 \cos\left(\frac{1}{x}\right) \le \mathbf{x^4}.$$

Since the red and blue terms have limit 0 as $x \to 0$, the squeeze theorem says $\lim_{x\to 0} x^4 \cos\left(\frac{1}{x}\right) = 0$.

(f) $\lim_{u\to -3} \frac{2-\sqrt{u^2-5}}{u+3}$ (Hint: multiply by the conjugate).

Solution. We multiply by 1 in the form $1 = \frac{2+\sqrt{u^2-5}}{2+\sqrt{u^2-5}}$, then factor, and terms cancel.

$$\begin{aligned}
\lim_{u\to -3} \frac{2-\sqrt{u^2-5}}{u+3} &= \lim_{u\to -3} \frac{2-\sqrt{u^2-5}}{u+3} \cdot \left(\frac{2+\sqrt{u^2-5}}{2+\sqrt{u^2-5}}\right) \\
&= \lim_{u\to -3} \frac{4-(u^2-5)}{(u+3)(2+\sqrt{u^2-5})} \\
&= \lim_{u\to -3} \frac{-(u-3)(u+3)}{(u+3)(2+\sqrt{u^2-5})} \\
&= \lim_{u\to -3} \frac{-(u-3)}{2+\sqrt{u^2-5}} \\
&= \frac{-(-3-3)}{2+\sqrt{(-3)^2-5}} \\
&= \frac{6}{4} \\
&= \frac{3}{2}
\end{aligned}$$

(g) $\lim_{t\to 0} t^2 2^{\sin\left(\frac{1}{t^2}\right)}$.

Solution. Since $-1 \le \sin(1/t^2) \le 1$ and $f(x) = 2^x$ is an increasing function,

$$2^{-1} \le 2^{\sin(1/t^2)} \le 2^1$$

and so

$$\mathbf{2^{-1}t^2} \le t^2 2^{\sin(1/t^2)} \le \mathbf{2^1 t^2}.$$

Since the red and blue terms approach 0 as $t \to 0$, the Squeeze Theorem says that

$$\lim_{t\to 0} t^2 2^{\sin\left(\frac{1}{t^2}\right)} = 0.$$

(h) $\lim_{t\to 7} \frac{\sqrt{t+2}-3}{t-7}$

Solution. Should be similar to (f). Your answer should be $\frac{1}{6}$.

(i) $\lim_{h \to 0} \frac{(4+h)^2 - 16}{h}$.

Solution. First expand $(4+h)^2 = 16 + 8h + h^2$. Then do the algebra carefully. Your answer should be 8.

(j) $\lim_{x \to 4} (x-4)^2 \sin\left(\frac{\cos(x)}{x-4}\right)$.

Solution. Answer is zero. Similar Squeeze Theorem argument to (e) and (g).

4. Is there a number a such that $\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$ exists? If so, find the value of a and the value of the limit.

Solution. The denominator factors as $x^2 + x - 2 = (x-2)(x+1)$. For the limit to exist, we need $(x-2)$ to be a factor in the numerator. (Why?) This happens if -2 is a root of the numerator, so let's see if we can make that happen. Plugging in $x = -2$ into the numerator: we want a such that

$$\begin{aligned} 3(-2)^2 + a(-2) + a + 3 &= 0 \\ 12 - 2a + a + 3 &= 0 \\ 15 - a &= 0 \\ a &= 15 \end{aligned}$$

If we choose $a = 15$, the limit exists. Check this yourself; try to figure out the limit with $a = 15$ plugged in. Factor the numerator and denominator, and you should see factors cancel and find that the limit exists.

Note. We used an important algebra fact here:

$$(x-a) \text{ is a factor of a polynomial } P(x) \iff P(a) = 0.$$

5. True or False.

(a) If $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$, then $\lim_{x \to 5} \frac{f(x)}{g(x)}$ does not exist. If the answer is false, give a counterexample (that is, an example that satisfies the hypothesis but not the conclusion).

Solution. False. Take e.g. $f(x) = x - 5$ and $g(x) = x - 5$. Then $\frac{f(x)}{g(x)}$ is equal to the constant function 1 with a hole at $x = 5$, and so $\lim_{x \to 5} \frac{f(x)}{g(x)}$ exists and is equal to 1.

(b) if $f(x) > 1$ for all $x > 0$ and if $\lim_{x \to 0} f(x)$ exists, then $\lim_{x \to 0} f(x) > 1$. If the answer is false, give a counterexample.

Solution. False. Counterexample: $f(x) = x + 1$.

(c) If $\lim_{x \to 6} f(x)g(x)$ exists, then the limit must be $f(6)g(6)$. If the answer is false, give a counterexample.

Solution. False. Counterexample: $f(x) = x - 6$ and $g(x) = \begin{cases} \frac{1}{x-6} & x \neq 6 \\ 4 & x = 6 \end{cases}$. The limit is equal to 1, but $f(6)g(6) = 0$. Notice that $h(x) = \frac{1}{x-a}$ is our standard go-to example when we need a function discontinuous at $x = a$.

(d) If $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to 0} g(x) = \infty$, then $\lim_{x \to 0} [f(x) - g(x)] = 0$. If the answer is false, give a counterexample.

Solution. False. Counterexample: $f(x) = \frac{2}{x^2}$ and $g(x) = \frac{1}{x^2}$. We have $f(x) - g(x) = \frac{1}{x^2}$, which diverges to $+\infty$ as $x \to 0$.

(e) $\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)$.

Solution. Combine into one fraction, factor, and cancel terms. We see:

$$\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to 4}\frac{2x-8}{x-4}=\lim_{x\to 4}\frac{2(x-4)}{x-4}=\lim_{x\to 4}2=2.$$

Notice that we cannot split up the limit as $\lim_{x\to 4}\frac{2x}{x-4}-\lim_{x\to 4}\frac{8}{x-4}$, since both of these limits are divergent.