Sept. 16. *Topics: Section 1.7 - The Precise Definition of a Limit, Section 1.8 - Continuity.*

One goal of today's worksheet is to help you understand and learn the following definition:
The Precise Definition of a Limit We say that $\lim_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

$\epsilon - \delta$ limit questions.

1. Write the given limit using the precise definition of limits.

   (a) $\lim_{x \to 1} x + 3 = 4$.

   Solution. For every $\epsilon > 0$, there is a $\delta > 0$ such that

   $$|x - 1| < \delta \quad \text{guarantees} \quad |(x + 3) - 4| < \epsilon.$$

   Since $|(x + 3) - 4| = |x - 1|$, we can choose $\delta = \epsilon$.

   (b) $\lim_{x \to -3} x^2 - 4 = 5$.

   Solution. For every $\epsilon > 0$, there is a $\delta > 0$ such that

   $$|x - (-3)| < \delta \quad \text{ensures} \quad |(x^2 - 4) - 5| < \epsilon.$$

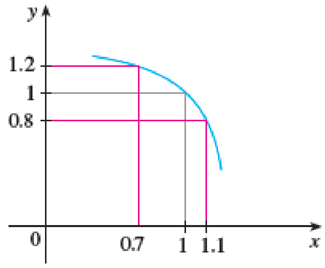
   (c) $\lim_{u \to -2\pi} \sin(u) = 0$.

   Solution. Similar. Compare to the examples above; we're carefully plugging in to the definition of $\lim_{x \to a} f(x)$ for specific choices of a and $f(x)$.

   (d) $\lim_{v \to b} g(v) = M$.

   Solution. Similar.

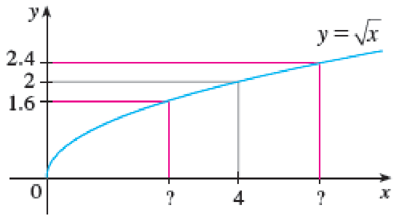
2. Use the given graph of f to find a number δ such that if $|x-1|<\delta$ then $|f(x)-1|<0.2$.



Solution. The picture shows that any x value in the range $(0.7, 1.1)$ guarantees $|f(x)-1|<0.2$. Any value of δ such that the interval $(x-\delta, x+\delta)$ stays inside $(0.7, 1.1)$ and contains the number 1 will work. The largest possible choice of δ is $\delta = 0.1$, which is the interval

$$(x-\delta, x+\delta) = (1-0.1, 1+0.1) = (0.9, 1.1).$$

3. Use the given graph of $f(x)=\sqrt{x}$ to find a number δ such that if $|x-4|<\delta$ then $|\sqrt{x}-2|<0.4$.



Solution. Like in the previous picture, we want to compute $f^{-1}(1.6)$ and $f^{-1}(2.4)$ to find the interval of x values that guarantee we stay in the range $(1.6, 2.4)$ surrounding the function value $f(4)=2$. The left-hand question mark is $f^{-1}(1.6)=1.6^2$ and the right-hand question mark is $f^{-1}(2.4)=2.4^2$. We want any value of δ such that the interval $(4-\delta, 4+\delta)$ is contained in $(1.6^2, 2.4^2)$, meaning

$$1.6^2 < 4-\delta < 4+\delta < 2.4^2.$$

We compute $1.6^2 = 2.56$ and $2.4^2 = 5.76$, so $\delta = 1$ is a safe bet. Since $|2.56-4| = 1.44$ and $|5.76-4| = 1.76$, the largest possible choice of δ is $\delta = 1.44$.

4. Look carefully at the following statements. Some of the statements are correct, and some have one or more issues. For each of the statements, decide whether it is correct or not. If it is not correct, identify the issue(s) and correct it (them).

   (a) We say that $\lim_{x \to b} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - b| < \epsilon$ then $|f(x) - L| < \delta$.

   Solution. Incorrect! The correct placement of ϵ and δ was swapped. Compare carefully to the true definition of a limit.

   Here's an example of a function that satisfies this definition but is not continuous: $f(x) = \sin(1/x)$. This is discontinuous at $x = 0$, but because $-1 \leq \sin(x) \leq 1$, then we could define $L = 0$ and $\delta = 2$ and satisfy this incorrect definition. Since

$$|f(x) - L| = |\sin(1/x) - 0| = |\sin(1/x)| \leq 1 < \delta,$$

   we can always guarantee the function error $|f(x) - L|$ is less than $\delta = 2$. A key principle of the $\epsilon - \delta$ definition of a limit $\lim_{x \to a} f(x) = L$ is that as as we move x close enough to a, we can guarantee $f(x)$ becomes within *any* small error ϵ, not one single choice of error δ.

   (b) We say that $\lim_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 \leq |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

   Solution. This is slightly off. Plugging in $x = a$ into this definition, we have $|x - a| = 0$, so $|x - a| < \delta$ always. Then the definition requires that $|f(a) - L| < \epsilon$ for any error ϵ, meaning that $f(a) = L$. But remember, it is not necessarily true that if $\lim_{x \to a} f(x) = L$, then $f(a) = L$, since $f(x)$ can have a hole at $x = a$. For example, consider the function

$$f(x) = \begin{cases} 1, & x \neq 0 \\ 8, & x = 0 \end{cases}.$$

   We have $\lim_{x \to 0} f(x) = 1$ but $f(0) \neq 1$.

   (c) We say that $\lim_{x \to a} f(x) = L$ if for every number $M > 0$ there is a number $N > 0$ such that if $0 < |x - a| < N$ then $|f(x) - L| < M$.

   Solution. Correct. We are just changing the names of the variables ϵ and δ to M and N here.

   (d) We say that $\lim_{u \to c} f(u) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |f(u) - c| < \delta$ then $|u - L| < \epsilon$.

   Solution. No. Flipping the roles of ϵ and δ gives an incorrect definition. For example, look at the constant function $f(u) = 1$. Then $\lim_{u \to 0} f(u) = 1$, of course, and $|f(u) - 1|$ is always equal to 0 for any input u; it is smaller than any error δ we like. But this does not necessarily imply that $|u - 1| < \epsilon$ for any error ϵ, since u can be any input.

5. Let $f(x) = 3x - 6$. Note that $\lim_{x\to 1} f(x) = -3$. Recall that the distance between two numbers a, b is given by $|b - a.|$.

(a) How close does x need to be to 1 so that $f(x)$ is as most 2 away from -3 (when $x \neq 1$)?

Solution. We will solve the inequality $|f(x) - (-3)| < 2$. Remember we are allowed to multiply and divide both sides of inequalities by positive numbers.

$$\begin{aligned} &|f(x) - (-3)| < 2 \\ \Leftrightarrow\ &|(3x - 6) - (-3)| < 2 \\ \Leftrightarrow\ &|3x - 3| < 2 \\ \Leftrightarrow\ &|x - 1| < \frac{2}{3}. \end{aligned}$$

Answer: x needs to be within distance $\frac{2}{3}$ of $x = 1$.

(b) How close does x need to be to 1 so that $f(x)$ is as most 0.3 away from -3 (when $x \neq 1$)?

Solution. Similarly, we will solve the inequality $|f(x) - (-3)| < 0.3$.

$$\begin{aligned} &|f(x) - (-3)| < 0.3 \\ \Leftrightarrow\ &|(3x - 6) - (-3)| < 0.3 \\ \Leftrightarrow\ &|3x - 3| < 0.3 \\ \Leftrightarrow\ &|x - 1| < 0.1. \end{aligned}$$

Answer: x needs to be within distance 0.1 of $x = 1$.

(c) How close does x need to be to 1 so that $f(x)$ is as most 0.02 away from -3 (when $x \neq 1$)?

Solution. Carry out the same procedure. You will find x needs to be within distance $\frac{0.02}{3} = \frac{2}{300}$ of $x = 1$.

(d) How close does x need to be to 1 so that $f(x)$ is as most ϵ away from -3 (when $x \neq 1$)?

(e) Find δ, in terms of ϵ, so that when $0 < |x - 1| < \delta$, we have $|f(x) - (-3)| < \epsilon$.

Solution. These are the same question rephrased. We will solve the inequality $|f(x) - (-3)| < \epsilon$.

$$\begin{aligned} &|f(x) - (-3)| < \epsilon \\ \Leftrightarrow\ &|(3x - 6) - (-3)| < \epsilon \\ \Leftrightarrow\ &|3x - 3| < \epsilon \\ \Leftrightarrow\ &|x - 1| < \frac{\epsilon}{3}. \end{aligned}$$

We found that x needs to be within distance $\frac{\epsilon}{3}$ of $x = 1$ in order to guarantee $|f(x) - (-3)| < \epsilon$. In other words, if we define $\delta = \frac{\epsilon}{3}$ then we have

$$|x - 1)| < \delta \quad \Longrightarrow \quad |f(x) - (-3)| < \epsilon.$$

6. Find a number δ such that if $|x-3|<\delta$, then $|3x-9|<0.01$.

Solution. Divide both sides of the inequality $|3x-9|<0.01$ by 3. Then we have $|x-3|<\frac{0.01}{3}=\frac{1}{300}$. So $\delta=\frac{1}{300}$ is a suitable number; our steps in reverse show that if $|x-3|<\frac{1}{300}$, then $|3x-9|<0.01$.

7. Given that $\lim_{x\to -2}(2x-7)=-11$, find δ so that when $0<|x+2|<\delta$, we have $|2x-7+11|<0.1$.

Solution. Break down the inequality $|2x-7+11|<0.1$.

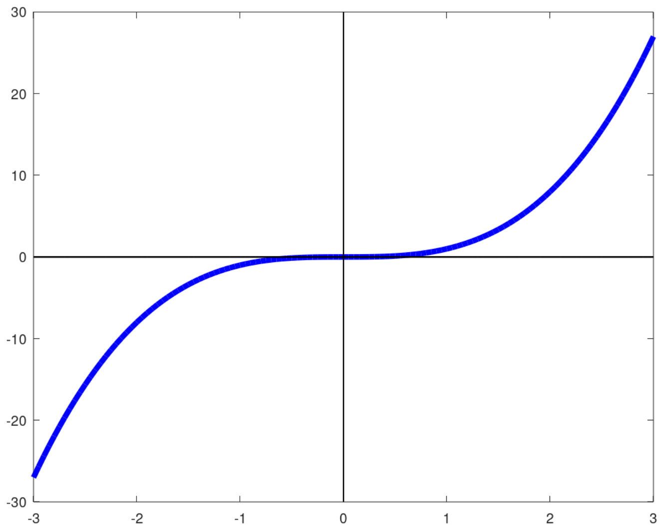
$$\begin{aligned}|2x-7+11|&<0.1\\ \Leftrightarrow |2x+4|&<0.1\\ \Leftrightarrow |x+2|&<0.05\\ \Leftrightarrow |x-(-2)|&<0.05\end{aligned}$$

So $\delta=0.05$ is a suitable choice.

8. Consider the function $f(x)=x^3$.

   (a) Determine $\lim\limits_{x\to 0} f(x)$ and sketch the graph of $f(x)$.

   Solution. The function is continuous at $x=0$: $\lim_{x\to 0} x^3=0^3=0$.



(b) For $\epsilon = 2$, determine δ such that $|x - 0| < \delta$ implies $|f(x) - L| < \epsilon$ by considering the problem graphically. Draw the "windows" on your graph.

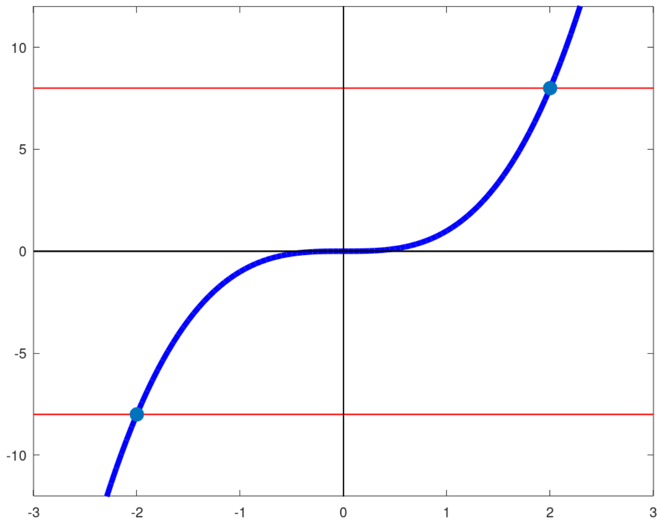
Solution. First, we know the value of the limit L. We have $L = \lim_{x\to 0} x^3 = 0$. So we want to find δ such that:

$$|x - 0| < \delta \implies |f(x) - L| < \epsilon$$

which is more specifically:

$$|x| < \delta \implies |x^3| < 2.$$

Visually, we want to know what range of x values keeps our function value $f(x) = x^3$ in the y-range $(-2, 2)$ bounded by the red lines $y = 2$ and $y = -2$. We can see this from the picture. The x-range corresponding to the y-range lies between the intersection dots.



The two intersection points are $((-2)^{1/3}, -2)$ and $(2^{1/3}, 2)$, which we find by solving the equations $x^3 = -2$ and $x^3 = 2$. Therefore the x-range is $(-2^{1/3}, 2^{1/3})$. Summarizing:

$$\text{If } -2^{1/3} < x < 2^{1/3} \text{ then } -2 < x^3 < 2.$$

In other words:

$$\text{If } |x| < 2^{1/3} \text{ then } |x^3| < 2.$$

In other words: for $\epsilon = 2$, $\delta = 2^{1/3}$, $L = 0$,

$$\text{If } |x - 0| < \delta \text{ then } |x^3 - L| < \epsilon.$$

(c) Repeat part (b) with $\epsilon = 1$. Make a prediction for how we can choose δ if we are given an arbitrary ϵ. (Hint: your δ should depend on ϵ).

Solution. If we repeat the same computations above, we find $\delta = 1$ instead of $\delta = 2^{1/3}$. We can do this algebraically too.

$$\begin{aligned} &|x^3| < \epsilon \\ \Leftrightarrow\ &|x|^3 < \epsilon \\ \Leftrightarrow\ &|x| < \epsilon^{1/3}. \end{aligned}$$

Therefore ϵ, we choose $\delta = \epsilon^{1/3}$ to ensure that

$$|x| < \delta \text{ implies } |x^3| < \epsilon.$$

This agrees with our previous answer. When $\epsilon = 2$ we had $\delta = \epsilon^{1/3} = 2^{1/3}$.

9. Let $f(x) = 5x + 3$. It is known that $\lim_{x \to 1} f(x) = 8$. Given $\epsilon > 0$ find $\delta > 0$, depending on ϵ, such that $|f(x) - 8| < \epsilon$ when $0 < |x - 1| < \delta$.

Solution. This is the same idea as # 5. We solve:

$$\begin{aligned} &|(5x + 3) - 8| < \epsilon \\ \Leftrightarrow\ &|5x - 5| < \epsilon \\ \Leftrightarrow\ &|x - 1| < \frac{\epsilon}{5}. \end{aligned}$$

So we choose $\delta = \frac{\epsilon}{5}$.

10. Let $f(x) = -4x+2$. It is known that $\lim_{x \to 2} f(x) = -6$. Given $\epsilon > 0$ find $\delta > 0$, depending on ϵ, such that $|f(x)+6| < \epsilon$ when $0 < |x - 2| < \delta$.

Solution. Similar: your answer should be $\delta = \frac{\epsilon}{4}$. [Note that if $\delta = \frac{\epsilon}{4}$ works, then a smaller number would also work. For example, $\delta = \frac{\epsilon}{90210}$ would work].

11. Let $f(x) = -7x-4$. It is known that $\lim_{x \to -1} f(x) = 3$. Given $\epsilon > 0$ find $\delta > 0$, depending on ϵ, such that $|f(x)-3| < \epsilon$ when $0 < |x + 1| < \delta$.

Solution. Same idea - do yourself carefully.

12. Let $f(x) = 3x+1$. It is known that $\lim_{x \to -2} f(x) = -5$. Given $\epsilon > 0$ find $\delta > 0$, depending on ϵ, such that $|f(x)+5| < \epsilon$ when $0 < |x + 2| < \delta$.

Solution. Same.

Continuity questions.

Plotting each of these functions will help you to identify what kind of discontinuity each function has. You will want to use the definition of continuity in terms of limits.

1. Show that $f(t) = \begin{cases} 4, & t \neq 1 \\ 5, & t = 1 \end{cases}$ is not continuous at $t = 1$. How can you "remove" this discontinuity?

Solution. We have

$$\lim_{t \to 1} f(t) = \lim_{t \to 1} 4 = 4$$

and $f(1) = 5$, so $f(t)$ is not continuous at $t = 1$ because

$$\lim_{t \to 1} f(t) \neq f(1).$$

If we change the value $f(1)$ to become $f(1) = 4$, then the function becomes continuous. For this reason, the discontinuity is called a removable discontinuity.

2. Show that $g(t) = \tan(t)$ is not continuous at $t = \pi/2$.

Solution. $g(t)$ is not continuous at $t = \pi/2$ because it is not defined at $t = \pi/2$.

3. Show that $h(t) = \begin{cases} \frac{t+2}{|t+2|}, & t \neq -2 \\ 1, & t = -2 \end{cases}$ is not continuous at $t = -2$.

Solution. We have

$$\lim_{t \to -2-} h(t) = \lim_{t \to -2-} \frac{t+2}{-(t+2)} = \lim_{t \to -2-} -1 = -1$$

and

$$\lim_{t \to -2+} h(t) = \lim_{t \to -2+} \frac{t+2}{(t+2)} = \lim_{t \to -2+} 1 = 1.$$

Since the corrresponding right-hand and left-hand limits disagree, $\lim_{t \to -2} h(t)$ does not exist, so $h(t)$ cannot be continuous at $t = -2$.

4. Show that $k(t) = \begin{cases} 1/t, & t \neq 0 \\ -1, & t = 0 \end{cases}$ is not continuous at $t = 0$.

Solution. Similar, $\lim_{t \to 0} k(t)$ does not exist; the right-hand and left-hand limits diverge differently.