Continuity questions.

1. State the definition of continuity.

   Solution. A function $f(t)$ is continuous at $t = t_0$ if

$$f(t_0) = \lim_{t \to t_0} f(t).$$

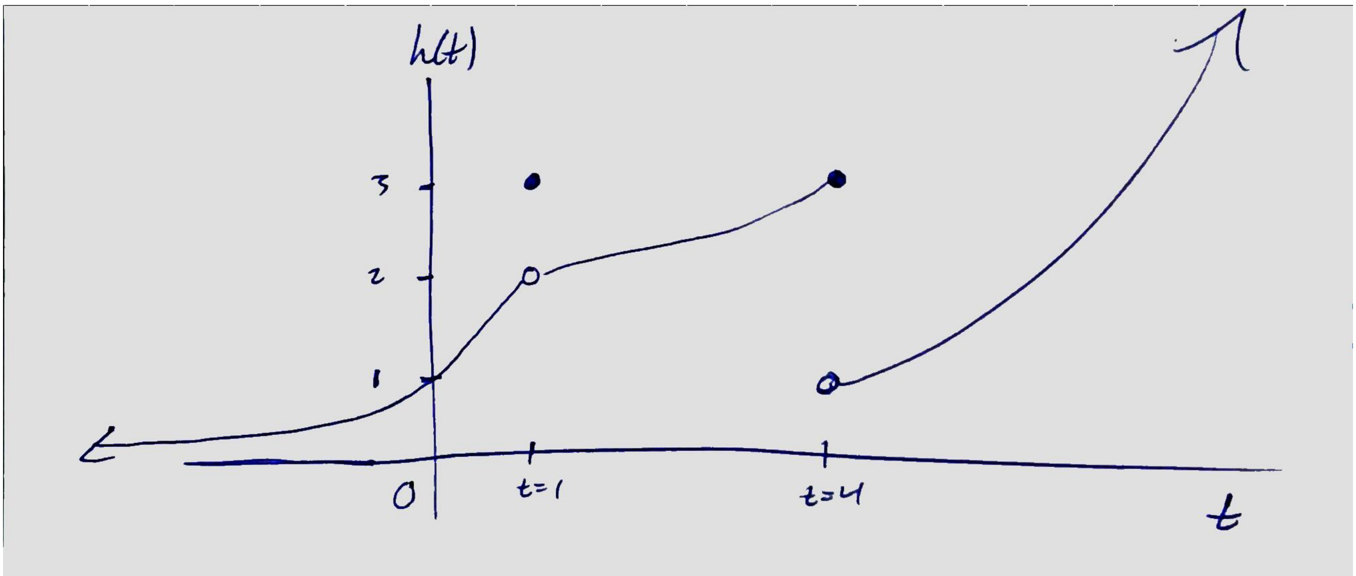
2. True or False: If $\lim_{x \to 0} f(x)$ exists, then $f(x)$ is continuous at $x = 0$. (If the statement is true, explain why. If the statement is false, come up with a counterexample.)

   Solution. False. Consider

$$f(x) = \begin{cases} 90210, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

3. Draw a graph of a function $h(t)$ that satisfies all of the following properties.
   (a) The domain of h is all real numbers and the range of h is all positive real numbers.
   (b) $h(t)$ is not continuous at $t = 1$ and at $t = 4$.
   (c) $\lim_{t \to 1^+} h(t) = 2$ and $\lim_{t \to 1^-} h(t) = 2$.
   (d) $\lim_{t \to 4^+} h(t) = 1$ and $\lim_{t \to 4^-} h(t) = 3$.

   One possible solution.



4. Consider the function $g(x) = \begin{cases} x & x < -2 \\ bx^2 & x \geq -2 \end{cases}$ where b is some constant.

   (a) Compute $\lim_{x \to -2^-} g(x)$.

   Solution. $\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x = -2$.

   (b) Compute $\lim_{x \to -2^+} g(x)$.

   Solution. $\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} bx^2 = b(-2)^2 = 4b$.

   (c) Compute $g(-2)$.

   Solution. $g(-2) = b(-2)^2 = 4b$.

   (d) For what value of b will $\lim_{x \to -2} g(x)$ exist?

   Solution. The limit exists, and the function is continuous, when the right-hand limit, left-hand limit, and function value, agree. This happens when $4b = -2$, in other words, when $b = \frac{-1}{2}$.

5. Let

$$g(x) = \begin{cases} ax + 2 & x < -1 \\ x^2 + b & -1 \leq x \leq 2 \\ 2x + 4 & x > 2. \end{cases}$$

Find the values of a and b that make g continuous everywhere.

Solution. The only potential issue we need to think about here is that the left-hand and right-hand limits agree at the inputs $x = -1$ and $x = 2$. The function $g(x)$ is continuous at $x = -1$ if

$$\begin{aligned} \lim_{x\to-1^-} g(x) &= \lim_{x\to-1^+} g(x) = g(-1) \\ \Leftrightarrow \lim_{x\to-1^-} ax + 2 &= \lim_{x\to-1^+} x^2 + b = 1 + b \\ \Leftrightarrow -a + 2 &= 1 + b = 1 + b \end{aligned}$$

The function $g(x)$ is continuous at $x = 2$ if:

$$\begin{aligned} \lim_{x\to2^-} g(x) &= \lim_{x\to2^+} g(x) = g(2) \\ \Leftrightarrow \lim_{x\to2^-} x^2 + b &= \lim_{x\to2^+} 2x + 4 = 8 \\ \Leftrightarrow 4 + b &= 8 = 8 \end{aligned}$$

Solving the second equation $4 + b = 8$, we find $b = 4$. Substituting $b = 4$ into the first equation $-a + 2 = 1 + b$, we find $-a + 2 = 5$, so $a = -3$. Therefore if $a = -3$, $b = 4$, the limit equalities written above hold, which guarantees that $g(x)$ is continuous at all inputs x.

6. Locate the discontinuities of the function $y(x) = \dfrac{4}{1+\cos(x)}$.

Solution. Since $\cos(x)$ is continuous, this function is continuous whenever it is defined, i.e. whenever $1+\cos(x) \neq 0$. The function is only discontinuous when $1 + \cos(x) = 0$, which happens when $x = \pi + 2\pi n$ for a whole number n.

7. Use the Intermediate Value Theorem to show that there exists c in $[0, 1]$ such that $f(c) = 0$, where $f(x) = -8x^4 + 2x^3 - x + 1$.

Solution. We compute $f(0) = 1$ and $f(1) = -6$. The Intermediate Value Theorem says that the value of $f(x)$ must hit the value 0 which lies between 1 and -6 for some input x in the interval $(0, 1)$.

8. Consider the function $f(x) = \dfrac{x^2-1}{x-1}$. How would you "remove the discontinuity" of f? In other words, how would you define $f(1)$ in order to make f continuous at 1?

Solution. Factoring and canceling as we have practiced before, we find:

$$f(x) = \begin{cases} x+1, & x \neq 1 \\ \text{undefined}, & x = 1 \end{cases}.$$

This becomes a continuous function if we define $f(1) = 2$.

9. Consider the function $f(x) = \dfrac{x^2+6x+8}{x+2}$. How would you "remove the discontinuity" of f?

Solution. Factor. For all inputs $x \neq -2$, we have

$$f(x) = \frac{(x+2)(x+4)}{x+2} = x + 4.$$

If we define $f(-2) = -2 + 4 = 2$, then the function becomes equal to the continuous function $g(x) = x + 4$ defined for all inputs.

10. Suppose $y = h(x)$ is the equation of a line. Find an equation for $h(x)$ if we are given that the following function $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} \dfrac{2x^2+6x+4}{3x^2-3} & x < -1 \\ h(x) & -1 \le x \le 3 \\ \dfrac{6}{x^2-9} - \dfrac{1}{x-3} & x > 3 \end{cases}$$

Solution. Factor and combine terms. We have for $x \neq \pm 1$:

$$\frac{2x^2+6x+4}{3x^2-3} = \frac{2(x^2+3x+2)}{3(x^2-1)} = \frac{2(x+2)(x+1)}{3(x-1)(x+1)} = \frac{2(x+2)}{3(x-1)}$$

and for $x \neq \pm 3$ we find a common denominator:

$$\begin{aligned} \frac{6}{x^2-9} - \frac{1}{x-3} &= \frac{6}{(x-3)(x+3)} - \frac{1}{x-3} \\ &= \frac{6}{(x-3)(x+3)} - \frac{x+3}{(x-3)(x+3)} \\ &= \frac{6-(x+3)}{(x-3)(x+3)} \\ &= \frac{-(x-3)}{(x-3)(x+3)} \\ &= \frac{-1}{x+3} \end{aligned}$$

Therefore we can rewrite the formulas defining $f(x)$ as:

$$f(x) = \begin{cases} \dfrac{2(x+2)}{3(x-1)}, & x < -1 \\ h(x), & -1 \le x \le 3 \\ \frac{-1}{x+3}, & x > 3 \end{cases}$$

Now in order for $f(x)$ to be continuous, we need to choose a continuous function $h(x)$ such that the following limit equalities hold:

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$$
$$\Rightarrow \frac{-1}{3} = h(-1)$$

and

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$$
$$\Rightarrow h(3) = \frac{-1}{6}.$$

There are many possible continuous functions $h(x)$ which have $h(-1) = \frac{-1}{3}$ and $h(3) = \frac{-1}{6}$. We will find the equation of a line which passes through the points $(-1, \frac{-1}{3})$ and $(3, \frac{-1}{6})$ using the point-slope equation. The slope between these points is $\frac{-1/6-(\frac{-1}{3})}{3-(-1)} = \frac{1}{24}$. Plugging into the point-slope formula, we have

$$h(x) = h(3) + m(x-3)$$
$$\Rightarrow \mathbf{h(x) = \frac{-1}{6} + \frac{1}{24}(x-3)}.$$

11. Show that there exists an intersection point between the graphs of $y = \sin(x)$ and $y = 4^{x/\pi}$ in the interval $\left(\frac{-3\pi}{2}, 0\right)$.

Solution. We perform a nifty trick. Let $f(x) = \sin(x) - 4^{x/\pi}$. Then

$$f(\frac{-3\pi}{2}) = \sin(\frac{-3\pi}{2}) - 4^{(\frac{-3\pi}{2})/\pi} = 1 - 4^{-3/2},$$

which is positive And

$$f(0) = \sin(0) - 4^{0/\pi} = -1,$$

which is negative. 0 is an intermediate value between these positive and negative numbers. The Intermediate Value Theorem says for some input x in $\left(\frac{-3\pi}{2}, 0\right)$, we have

$$f(x) = \sin(x) - 4^{x/\pi} = 0,$$

meaning that

$$\sin(x) = 4^{x/\pi},$$

in other words, we have an intersection point in the interval $\left(\frac{-3\pi}{2}, 0\right)$.

12. Suppose f is continuous on $[2, 8]$ and the only solution of the equation $f(x) = 4$ are $x = 3$ and $x = 7$. If $f(4) = 6$, explain why $f(5) > 4$.

Solution. If it were true that $f(5) \leq 4$, then also it must be true that $f(5) < 4$, since we assumed $f(5) \neq 4$. Then since $f(4) = 6$, the Intermediate Value Theorem says that $f(x) = 4$ for some x between 4 and 5, since 4 is an intermediate value between $f(5)$ (which is less than 4) and 6. But we assumed that $f(x) = 4$ only when $x = 3$ and $x = 7$, which is a problem. So it couldn't be true in the first place that $f(5) \leq 4$, so $f(5) > 4$.

13. Find the equation of the line through the points $(2, 4)$ and $(1, -2)$.

Solution. Let's plug in to the point-slope formula $f(x) = f(x_0) + m(x - x_0)$ with $x_0 = 1$. Then $f(x_0) = f(1) = -2$, and the slope m is $m = \frac{-2-4}{1-2} = 6$. So:

$$f(x) = -2 + 6(x - 1).$$

14. Let $f(x) = \sqrt{x}$.

(a) Find the slope of the line through the points $(4, f(4))$ and $(9, f(9))$.

Solution. The slope is:

$$\frac{f(9) - f(4)}{9 - 4} = \frac{f(9) - f(4)}{5} = \frac{3 - 2}{5} = \frac{1}{5}.$$

(b) Find the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

Solution.

$$\frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a} = \frac{1}{\sqrt{b} + \sqrt{a}}.$$

Derivative questions.

1. Let f be a function. Find the equation of the slope of the secant line that passes through the points $(a, f(a))$ and $(a+h, f(a+h))$.

   Solution. By "secant line," we're just emphasizing that the line is related to the function $f(x)$ itself. We carry out the normal procedure for finding the slope of the line that passes through two given points.

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}.$$

2. Let f be a function. Find the equation of the slope of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$.

   Solution.

$$\frac{f(b)-f(a)}{b-a}.$$

3. Let $f(x) = 2x^2$.

   (a) Find the slope of the line through the points $(1, f(1))$ and $(2, f(2))$.

   Solution. We have $(1, f(1)) = (1, 2)$ and $(2, f(2)) = (2, 8)$. The slope of the line passing through these two points is 6.

   (b) Find the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

   Solution.

$$\frac{f(b)-f(a)}{b-a} = \frac{2b^2-2a^2}{b-a} = \frac{2(b-a)(b+a)}{b-a} = 2(b+a).$$

   (c) Compute $\lim_{b\to 1} \frac{f(b)-f(1)}{b-1}$.

   Solution. Substituting $a = 1$ into the formula we just computed:

$$\lim_{b\to 1} \frac{f(b)-f(1)}{b-1} = \lim_{b\to 1} 2(b+1) = 2(1+1) = 4.$$

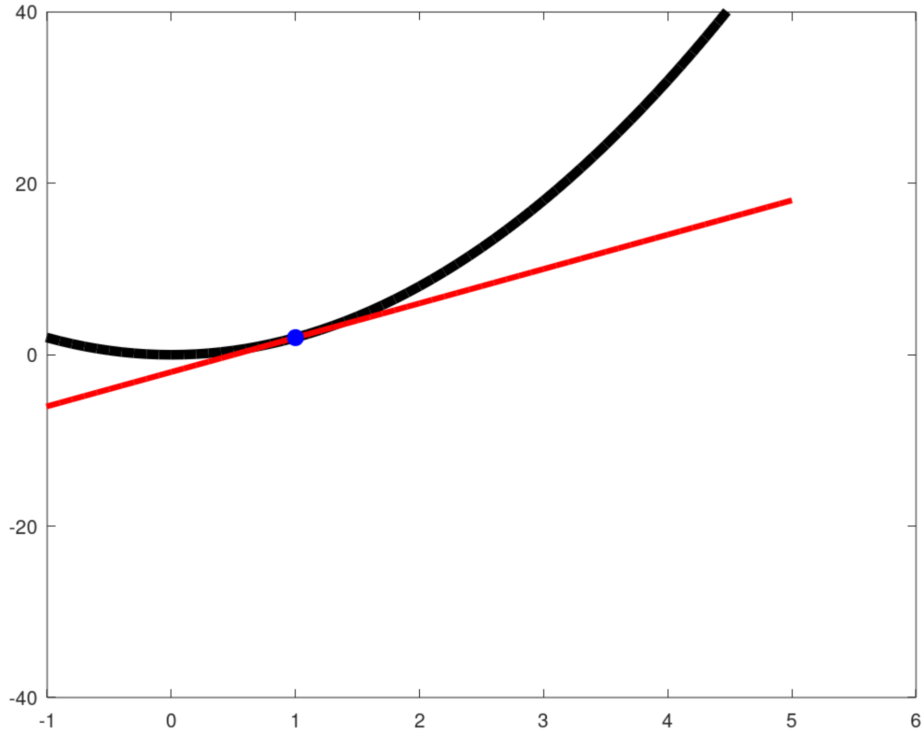
   (d) Write the equation of the line tangent to $f(x) = 2x^2$ at $x = 1$.

   Solution. Part (c) above tells us the tangent slope is equal to 4. The tangent line has slope 4 and passes through the point $(1, f(1)) = (1, 2)$. Therefore, using the point-slope equation of a line, we find the equation of the tangent line at $x = 1$ is:

$$T(x) = 4(x-1) + 2.$$

(e) Sketch the function $f(x) = 2x^2$ and the tangent line to the function at $x = 1$.

Solution. Notice the function "just touches" the graph of $f(x)$ at $x = 1$.



4. Use the definition of the derivative to find the derivative of the function $f(x) = 3x^2 + 4$ at the point $x = 2$.

Solution. We insert this specific function $f(x)$ into the definition of the derivative carefully.

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\big(3(2+h)^2 + 4\big) - \big(3(2)^2 + 4\big)}{h} \\ &= \lim_{h \to 0} \frac{3(4 + 4h + h^2) + 4 - 16}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} \\ &= \lim_{h \to 0} (12 + 3h) \\ &= 12. \end{aligned}$$

5. Use the definition of the derivative to find the derivative of the function $f(x) = \dfrac{1}{x-2}$ at the point $x = -1$.

Solution. Notice the technique of finding a common denominator when two rational functions are added together.

$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1}{(-1+h)-2} - \frac{-1}{3}}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \left(\frac{1}{h-3} + \frac{1}{3} \right) \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \left(\frac{3}{3(h-3)} + \frac{(h-3)}{3(h-3)} \right) \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{3 + (h-3)}{3(h-3)} \\ &= \lim_{h \to 0} \frac{1}{h} \cdot \frac{h}{3(h-3)} \\ &= \lim_{h \to 0} \frac{1}{3(h-3)} \\ &= \frac{1}{3(-3)} \\ &= \frac{-1}{9}. \end{aligned}$$