1. Let f be a function. Find the equation of the slope of the secant line that passes through the points $(a, f(a))$ and $(a+h, f(a+h))$.

   Solution. By "secant line," we're just emphasizing that the line is related to the function $f(x)$ itself. We carry out the normal procedure for finding the slope of the line that passes through two given points.

   $$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}.$$

   Read back to yourself how this explains why the derivative is the slope of a tangent line.

2. Let f be a function. Find the equation of the slope of the secant line that passes through the points $(a, f(a))$ and $(b, f(b))$.

   Solution.

   $$\frac{f(b) - f(a)}{b - a}.$$

3. Let $f(x) = 2x^2$.

   (a) Find the slope of the line through the points $(1, f(1))$ and $(2, f(2))$.

   Solution. We have $(1, f(1)) = (1, 2)$ and $(2, f(2)) = (2, 8)$. The slope of the line passing through these two points is 6.

   (b) Find the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

   Solution.

   $$\frac{f(b) - f(a)}{b - a} = \frac{2b^2 - 2a^2}{b - a} = \frac{2(b-a)(b+a)}{b - a} = 2(b+a).$$

   (c) Compute $\lim_{b \to 1} \frac{f(b) - f(1)}{b - 1}$.

   Solution. Substituting $a = 1$ into the formula we just computed:

   $$\lim_{b \to 1} \frac{f(b) - f(1)}{b - 1} = \lim_{b \to 1} 2(b+1) = 2(1+1) = 4.$$

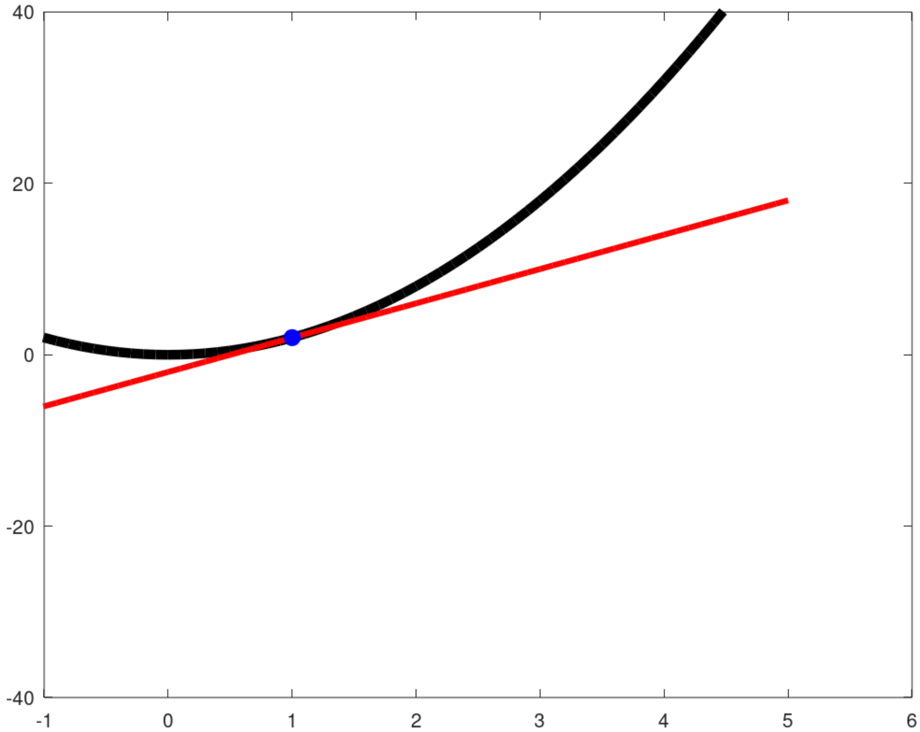
   (d) Write the equation of the line tangent to $f(x) = 2x^2$ at $x = 1$.

   Solution. Part (c) above tells us the tangent slope is equal to 4. The tangent line has slope 4 and passes through the point $(1, f(1)) = (1, 2)$. Therefore, using the point-slope equation of a line, we find the equation of the tangent line at $x = 1$ is:

   $$T(x) = 4(x-1) + 2.$$

(e) Sketch the function $f(x) = 2x^2$ and the tangent line to the function at $x = 1$.

Solution. Notice the function "just touches" the graph of $f(x)$ at $x = 1$.



4. Use the definition of the derivative to find the derivative of the function $f(x) = 3x^2 + 4$ at the point $x = 2$.

Solution. We insert this specific function $f(x)$ into the definition of the derivative carefully.

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{\big(3(2+h)^2 + 4\big) - \big(3(2)^2 + 4\big)}{h} \\ &= \lim_{h \to 0} \frac{3(4 + 4h + h^2) + 4 - 16}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} \\ &= \lim_{h \to 0} (12 + 3h) \\ &= 12. \end{aligned}$$

5. Use the definition of the derivative to find the *derivative function* $f'(x)$ corresponding to $f(x) = \dfrac{1}{x-2}$. Now what is $f'(-1)$? $f'(10)$?

Solution. Notice the technique of finding a common denominator when two rational functions are added together.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)-2} - \frac{1}{x-2}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{1}{x+h-2} - \frac{1}{x-2} \right) \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{(x-2)}{(x+h-2)(x-2)} - \frac{(x+h-2)}{(x+h-2)(x-2)} \right) \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x-2)-(x+h-2)}{(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-2)(x-2)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-2)(x-2)} \\
&= \frac{-1}{(x-2)(x-2)} \\
&= \frac{-1}{(x-2)^2}.
\end{aligned}$$

Now we have a formula we can use: $f'(x) = \frac{-1}{(x-2)^2}$. We have $f'(-1) = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$. And $f'(10) = \frac{-1}{(10-2)^2} = \frac{-1}{64}$.

6. Use the definition of the derivative to find the derivative $f'(6)$ where $f(x) = \sqrt{x-4}$.

Solution.

$$\begin{aligned}
f'(6) &= \lim_{h\to 0} \frac{f(6+h)-f(6)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(6+h)-4} - \sqrt{6-4}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \left(\frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} \right) \\
&= \lim_{h\to 0} \frac{(2+h)-2}{h(\sqrt{2+h} + \sqrt{2}} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} \\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

7. Using the definition of the derivative, compute $g'(1)$ where $g(x) = x^2 - 3x$.

Answer. -1. Same method as the last few problems.

8. Consider the function $y = \frac{3}{2+x}$.

(a) Using the definition of the derivative, compute the slope of the tangent line to the function $y = \frac{3}{2+x}$ at the point $(-1, 3)$.

Solution. This means computing $y'(-1)$. The answer is $y'(-1) = -3$.

(b) Find the equation of the line tangent to $y(x)$ at $x = -1$.

Solution. Using the tangent slope $y'(-1) = -3$ and the point-slope formula, the tangent line is

$$T(x) = -3(x+1) + 3.$$

9. Let $f(x) = x^3 - 3x + 1$. Using the definition of the derivative, compute the slope of the tangent line at the point $(a, f(a))$. Where is the tangent line horizontal? Use this to sketch a graph of $y = f(x)$.

Partial solution. When you're inputting into the definition of the derivative $f'(a)$ carefully, you'll have a term $(a+h)^3$. You will expand that out, as:

$$\begin{aligned}(a+h)^3 &= (a+h)^2(a+h) \\ &= (a^2 + 2ah + h^2)(a+h) \\ &= a^3 + 2a^2h + ah^2 + a^2h + 2ah^2 + h^3 \\ &= a^3 + 3a^2h + 3ah^2 + h^3\end{aligned}$$

Proceed carefully as before. The answer is $f'(a) = 3a^2 - 3$.

10. Suppose the position of a car is given by the function $s(t) = t - t^2$ for $t \geq 0$.

(a) Find the average velocity of the car from $t = 0$ to $t = \frac{1}{2}$.
(b) Find the instantaneous velocity of the car at time $t = 1$.
(c) At what time is the car stopped?

Partial solution.

(a) The average velocity over this interval is

$$\frac{v(1/2) - v(0)}{(1/2) - 0} = \frac{1/4 - 0}{1/2} = \frac{1}{2}.$$

(b) The instantaneous velocity at $t = 1$ is $s'(1)$, which if you compute carefully you will find is equal to -1.

(c) "Being stopped" means "instantaneous velocity is equal to 0." The car is stopped at $t = 1$, and computing the derivative yields $s'(t) = 1 - 2t$ at any time t. Therefore solving $1 - 2t = 0$, we see that at $t = 1/2$, we have $s'(t) = 0$. The car is stopped at $t = 1/2$.

11. Show that $r(t) = \begin{cases} 1, & t < 0 \\ t+1, & t \geq 0 \end{cases}$ is not differentiable at $t = 0$. What does it mean for $r(t)$ to not be differentiable at $t = 0$?

Solution. $r(t)$ not being differentiable at $t = 0$ means that the limit

$$\lim_{h \to 0} \frac{r(0+h) - r(0)}{h}$$

does not exist. Let us show it doesn't exist. It doesn't exist because the corresponding right-hand and left-hand limits disagree. Using the different formulas defining $r(t)$ for t to the right and left of $t = 0$:

$$\begin{aligned} \lim_{h \to 0-} \frac{r(0+h) - r(0)}{h} &= \lim_{h \to 0-} \frac{r(h) - 1}{h} \\ &= \lim_{h \to 0-} \frac{1 - 1}{h} \\ &= \lim_{h \to 0-} 0 \\ &= 0. \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0+} \frac{r(0+h) - r(0)}{h} &= \lim_{h \to 0+} \frac{r(h) - 1}{h} \\ &= \lim_{h \to 0+} \frac{(h+1) - 1}{h} \\ &= \lim_{h \to 0-} 1 \\ &= 1. \end{aligned}$$

Therefore the derivative limit does not exist; the function is not differentiable at $t = 0$.