1. Let $f(x) = 2x^2 + x + 1$.

   (a) Compute the derivative $f'(x)$ using the definition of the derivative.

   **Solution.**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\big(2(x+h)^2 + (x+h) + 1\big) - \big(2x^2 + x + 1\big)}{h} \\ &= \lim_{h\to 0} \frac{(2(x^2 + 2xh + h^2) + x + h + 1 - (2x^2 + x + 1)}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 + h}{h} \\ &= \lim_{h\to 0} (4x + 2h + 1) \\ &= \lim_{h\to 0} 4x + 1. \end{aligned}$$

   (b) Your friend says that the equation for the tangent line to $f(x)$ at the point $(1, 4)$ is

$$y - 4 = (4x + 1)(x - 1).$$

   What did they do wrong?

   **Solution.** Your friend has strayed far from the flock. This is not even an equation of a line; it is a quadratic equal to $y = 4x^2 - 3x + 3$.

2. Use the definition of the derivative to find the derivative $f'(x)$ where $f(x) = \sqrt{x+1}$.

**Solution.** Ye olde conjugate-square-root method. Multiplying by one creatively helps eliminate some square roots and induce cancellation.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h} \cdot \left( \frac{\sqrt{x+h+1} + \sqrt{x+1}}{\sqrt{x+h+1} + \sqrt{x+1}} \right) \\ &= \lim_{h\to 0} \frac{(x+h+1) - (x+1)}{h(\sqrt{x+h+1} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+1} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+1} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+1} + \sqrt{x+1}} \\ &= \frac{1}{2\sqrt{x+1}}. \end{aligned}$$

3. Let $f(x) = x + |x|$. What is $f'(c)$ for $c > 0$?. What is $f'(c)$ for $c < 0$?. What about $f'(0)$?

**Solution.** Let's see.

When $c > 0$, then $|c| = c$, and $|c+h| = c+h$ when $h$ is close to zero. Therefore:

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{((c+h) + |c+h|) - (c+|c|)}{h} \\ &= \lim_{h\to 0} \frac{((c+h) + (c+h)) - (c+c)}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} \\ &= \lim_{h\to 0} 2 \\ &= 2. \end{aligned}$$

When $c < 0$, then $|c| = -c$, and $|c+h| = -(c+h)$ when $h$ is close to zero. Therefore:

$$\begin{aligned} f'(c) &= \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \\ &= \lim_{h\to 0} \frac{((c+h) + |c+h|) - (c+|c|)}{h} \\ &= \lim_{h\to 0} \frac{((c+h) - (c+h)) - (c-c)}{h} \\ &= \lim_{h\to 0} \frac{0}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0. \end{aligned}$$

When $c = 0$, then $|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$. We will show that the limit $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist by showing that the corresponding right-hand and left-hand limits are not equal.

We have

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{(h+|h|) - 0}{h} \\ &= \lim_{h\to 0^+} \frac{h+h}{h} \\ &= \lim_{h\to 0^+} 2 \\ &= 2 \end{aligned}$$

and

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{(h+|h|) - 0}{h} \\ &= \lim_{h\to 0^-} \frac{h-h}{h} \\ &= \lim_{h\to 0^-} 0 \\ &= 0. \end{aligned}$$

We conclude that the derivative does not exist.

4. Is the function

$$f(x) = \begin{cases} 0 & : x \leq 0 \\ x^2 & : x > 0 \end{cases}$$

continuous at $x = 0$? Is it differentiable at $x = 0$?

**Solution.** The function is continuous $x = 0$, because

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$

and

$$f(0) = 0.$$

Since these three quantities all agree with each other, the function is continuous at $x = 0$.

The function is differentiable at $x = 0$ also. We have

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{0 - 0}{h} = 0$$

and

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 - 0}{h} = \lim_{h \to 0} h = 0.$$

Therefore the limit $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ exists and is equal to 0, since the corresponding right-hand and left-hand limits are both equal to 0. We conclude that $f$ is differentiable at $x = 0$, and $f'(0) = 0$.

Technically, we could have saved ourselves a little work by showing $f(x)$ is differentiable at 0 first. If we can show $f(x)$ is differentiable at $x = 0$, we can automatically conclude that it is continuous at $x = 0$.

5. For which values of $a$ and $b$ is the following function differentiable at $x = 1$? Sketch a graph for those values of $a$ and $b$.

$$f(x) = \begin{cases} ax^2 + b & : x < 1 \\ x - x^2 & : x \geq 1 \end{cases}$$

**Solution - abbreviated algebra.** Let's make sure first that the function is continuous at $x = 1$. For this continuity to hold, we need: $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = f(1) = 0$. We have:

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (ax^2 + b) = \lim_{x\to 1-} a(-1)^2 + b = a + b$$

and

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x - x^2 = 0.$$

Therefore if $a + b = 0$, we guarantee that $f(x)$ is continuous at $x = 1$. We don't have enough information to identify the two unknowns $a$ and $b$ yet.

We also need to make sure $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ exists. We'll compare the corresponding right-hand and left-hand limits. Let's assume that a solution $(a, b)$ exists, which we know needs to have $a + b = 0$. A $(a + b)$ term will crop up in our computations, and we will use the equation $a + b = 0$ to deal with this term. We compute:

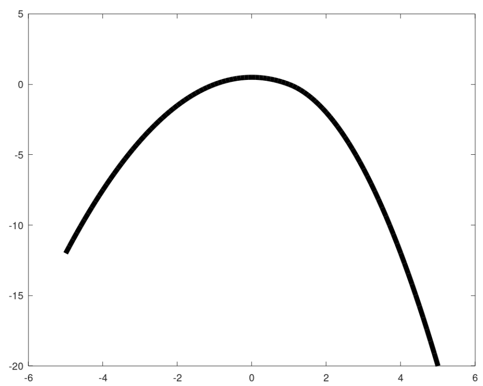
$$\begin{aligned} &\lim_{h\to 0-} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0-} \frac{(a(1+h)^2 + b) - 0}{h} \\ &= \lim_{h\to 0-} \frac{(a+b) + 2ah + h^2}{h} \\ &= \lim_{h\to 0-} (2a + h) \\ &= 2a \end{aligned}$$

and

$$\begin{aligned} &\lim_{h\to 0+} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0+} \frac{((1+h) - (1+h)^2) - (0)}{h} \\ &= \lim_{h\to 0+} \frac{-h - h^2}{h} \\ &= \lim_{h\to 0+} (-1 - h) \\ &= -1 \end{aligned}$$

Therefore we need $2a = -1$ if we want the derivative $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$ to exist. The derivative limit exists when $a = \frac{-1}{2}$. And $a + b = 0$ can be solved now with $\frac{-1}{2} + b = 0 \Rightarrow b = \frac{1}{2}$. Looking at our previous computations, we see that $f(x)$ *is differentiable at* $x = 1$ *when* $a = \frac{-1}{2}$ *and* $b = \frac{1}{2}$.

Here is the graph.



6. Let $f(x) = x + 2$, $g(x) = 2x - 1$.

(a) Compute $f'(x)$ and $g'(x)$.

**Partial solution.**

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1. \end{aligned}$$

By a similar computation, $g'(x) = 2$. The derivative of a linear function is always equal to the line's slope.

(b) Compute $[f(x)g(x)]'$. How does it compare to $f'(x)g'(x)$?

**Solution - abbreviated algebra.** We have $f(x)g(x) = (x+2)(2x-1) = 2x^2 + 3x - 2$. We compute:

$$\begin{aligned} (f(x)g(x))' &= \lim_{h\to 0} \frac{\big(2(x+h)^2 + 3(x+h) - 2\big) - (2x^2 + 3x - 2)}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2}{h} \\ &= \lim_{h\to 0} 4x + 2h \\ &= 4x. \end{aligned}$$

Moral: $f'(x)g'(x) \neq (f(x)g(x))'$, in other words, derivatives do not distribute across multiplication. In this example, $f'(x)g'(x)$ is a constant but $(f(x)g(x))'$ is not.

7. Let $f, g$ be functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = -5$, and $g'(2) = 2$. Use differentiation rules to find $h'(2)$ for

(a) $h(x) = 3f(x) - g(x)$

**Solution.**

$$\begin{aligned} h'(x) &= (3f(x) - g(x))' \\ &= (3f(x))' - (g(x))' \\ &= 3f'(x) - g'(x). \end{aligned}$$

Therefore $h'(2) = 3f'(2) - g'(2) = 3(-1) - 2 = -5$.

(b) $h(x) = f(x)g(x)$

**Solution.** Using the product rule:

$$h'(2) = f'(2)g(2) + f(2)g'(2) = (-1)(-5) + (3)(2) = 11.$$

(c) $h(x) = \frac{1}{f(x)}$

**Solution.** Quotient rule. Using the fact that the derivative of a constant function is constant, we have:

$$h'(2) = \frac{-f'(2)}{f(x)^2} = \frac{-(-1)}{3^2} = \frac{1}{9}.$$

(d) $h(x) = \frac{g(x)}{f(x)}$

**Solution.**

$$h'(2) = \frac{f(2)g'(2) - g(2)f'(2)}{f(2)^2} = \frac{(3)(2) - (-5)(-1)}{(3)^2} = \frac{1}{9}.$$

8. Compute the derivatives of the following functions:

(a) $f(x) = 4\pi^2$

**Solution.** The derivative of a constant function is zero.

$$\begin{aligned} f'(x) &= \lim_{y\to x} \frac{f(y)-f(x)}{y-x} \\ &= \lim_{y\to x} \frac{4\pi^2 - 4\pi^2}{y-x} \\ &= \lim_{y\to x} 0 \\ &= 0. \end{aligned}$$

(b) $f(x) = 8\sqrt{x}\cos(x)$.

**Solution.** Power rule, product rule, and our formula for the derivative of cosine. First let's rewrite $f(x) = 8x^{1/2}\cos(x)$. Now we can see:

$$f'(x) = 8 \cdot \frac{1}{2}x^{-1/2}\cos(x) - 8x^{1/2}\sin(x) = 4x^{-1/2}\cos(x) - 8x^{1/2}\sin(x).$$

(c) $f(x) = x^3 + 2x + 4$

**Solution.** Distributing the derivative across sums of functions and using the power rule, we have:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3) + \frac{d}{dx}(2x) + \frac{d}{dx}(4) \\ &= 3x^2 + 2 + 0 \\ &= 3x^2 + 2. \end{aligned}$$

(d) $f(x) = \frac{x^2-2x+1}{\sqrt{x}}$

**Solution.** Distribute.

$$\begin{aligned} f'(x) &= \left(\frac{x^2-2x+1}{x^{1/2}}\right)' \\ &= \left(x^{3/2} - 2x^{1/2} + x^{-1/2}\right)' \\ &= \left(x^{3/2}\right)' - 2\left(x^{1/2}\right)' + \left(x^{-1/2}\right)' \\ &= \frac{3}{2}x^{1/2} - 2\cdot\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} \\ &= \frac{3}{2}x^{1/2} - x^{-1/2} - \frac{1}{2}x^{-3/2}. \end{aligned}$$

(e) $f(x) = \frac{2x-1}{3x+2}$

**Solution.** We use the quotient rule, and the fact that the derivative of a linear function $f(x) = mx + b$ is the constant function $f'(x) = m$.

$$\begin{aligned} f'(x) &= \frac{(3x+2)(2x-1)' - (2x-1)(3x+2)'}{(3x+2)^2} \\ &= \frac{(3x+2)2 - (2x-1)3}{(3x+2)^2} \\ &= \frac{7}{(3x+2)^2}. \end{aligned}$$

(f) Compute $g'(r)$ and $g''(r)$ for $g(r) = \left(\frac{1}{r^2} - \frac{3}{r^4}\right)(r + 5r^3)$.

**Solution.** Let's distribute first.

$$g(r) = \frac{1}{r} + 5r - \frac{3}{r^3} - \frac{15}{r} = 5r - 14r^{-1} - 3r^{-3}.$$

Now it's easier to use the power rule.

$$g'(r) = 5 + 14r^{-2} + 9r^{-4}.$$

And

$$g''(r) = (g'(r))' = -28r^{-3} - 36r^{-5}.$$

(g) Find the first and second derivatives of $f(t) = (1 - 7t)^2$.

**Solution.** Distribute.

$$f(t) = 1 - 14t + 49t^2.$$

Now $f'(t) = -14 + 98t$ and $f''(t) = 98$.

9. Suppose $f(x)$ is a function which passes through the point $(4,3)$, and that the line tangent to $y = f(x)$ at $(4,3)$ also passes through the point $(0,2)$.

   (a) Sketch the tangent line along with a *possible* graph of $f(x)$ (make sure to label the two given points).

   (b) Find an equation of the tangent line you drew.

   **Solution.** We have all the information we need to find the equation of the tangent line $T(x)$. And there is an easy possible choice of $f(x)$: we can simply take $f(x) = T(x)$. Then $f(x)$ will be its own tangent line. The tangent line has slope $\frac{2-3}{0-4} = \frac{1}{4}$, so using the point-slope equation of a line, we have $f(x) = T(x) = \frac{1}{4}(x-0)+2 = \frac{1}{4}x+2$.

   Draw a picture of the line $f(x) = \frac{1}{4}x + 2$ and try to explain to yourself why at any $x$-value, the tangent line of $f(x)$ at $x$ is equal to $f(x)$ itself.

   (c) What is $f(4)$? What is $f'(4)$?

   **Solution.** Using the given information, $f(4) = 3$, and $f'(4) = \frac{1}{4}$, since the value of the derivative $f'(4)$ is equal to the tangent slope of $f(x)$ at $x = 4$.

10. Let $f(x) = \dfrac{x-1}{x+1}$. What is $(x+1)\cdot f(x)$? Can you use this to come up with a formula for $f'(x)$ without using the quotient rule?

**Solution.** We have $(x+1)\cdot f(x) = x-1$ (when $x =\neq -1$). Using the product rule and solving for $f'(x)$, we have:

$$\begin{aligned}
\frac{d}{dx}\left((x+1)\cdot f(x)\right) &= \frac{d}{dx}(x-1) \\
(x+1)'\cdot f(x) + (x+1)f'(x) &= 1 \\
f(x) + (x+1)f'(x) &= 1 \\
\frac{x-1}{x+1} + (x+1)f'(x) &= 1 \\
(x+1)f'(x) &= 1 - \frac{x-1}{x+1} \\
f'(x) &= \frac{1}{x+1} - \frac{x-1}{(x+1)^2}.
\end{aligned}$$