1. Compute the derivatives of the following functions.

   (a) $f(x) = x\sin(x)$.

   (b) $g(t) = \dfrac{4t^2}{\cos(t)}$.

   (c) $h(z) = \dfrac{\sin(z)}{3z^2+\pi}$.

   (d) $y(t) = \sqrt{t}\cos(t)$.

   (e) $f(x) = \tan(x)$.

   (f) $y(x) = \sec(x)$.

   (g) $g(v) = v^3\sec(v)$.

   (h) $h(s) = s^2\cos(s)\sin(s)$.

   (i) $f(x) = x^2\sin(x)\cos(x)\tan(x)$.

   (j) $h(t) = \sin^3(t)$.

   Solution - some abbreviated algebra.

   (a) Product rule. $f'(x) = \sin(x) + x\cos(x)$.

   (b) Quotient rule. $g'(t) = \frac{8t\cos(t)+4t^2\cos(t)}{\cos^2(t)}$.

   (c) Quotient rule. $h'(z) = \frac{(3z^2+\pi)\cos(z)-6z\sin(z)}{(3z^2+\pi)^2}$.

   (d) Product rule. Writing $y(t) = t^{1/2}\cos(t)$, then $y'(t) = (1/2)t^{-1/2}\cos(t) - t^{1/2}\sin(t)$.

   (e) Quotient rule. Writing $f(x) = \frac{\sin(x)}{\cos(x)}$, then

   $$f'(x) = \frac{-\cos(x)^2+\sin(x)^2}{\cos(x)^2} = -1+\tan(x)^2 = \sec(x)^2.$$

   (f) Quotient rule. Writing $y(x) = \frac{1}{\cos(x)}$, then

   $$y'(x) = \frac{\cos(x)\cdot 0+\sin(x)}{\cos(x)^2} = \frac{\sin(x)}{\cos(x)^2} = \tan(x)\sec(x).$$

   (g) Product rule. Using the formula we just figured out, $\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$, we have

   $$g'(v) = 3v^2\sec(v) + v^3\sec(v)\tan(v).$$

   (h) Product rule ($2\times$). Let's view $h(s)$ as the product $h(s) = s^2\cos(s)\sin(s)$ and use the product rule on the red part and blue parts.

   $$\begin{aligned} h'(s) &= (s^2)'(\cos(s)\sin(s)) + s^2(\cos(s)\sin(s))' \\ &= 2s\cos(s)\sin(s) - s^2\sin(s)^2 + s^2\cos(s)^2. \end{aligned}$$

   (i) Multi-product rule. Using the formula we figured out $\frac{d}{dx}(\tan(x)) = \sec(x)^2$, we have:

   $$f'(x) = 2x\sin(x)\cos(x)\tan(x) + x^2\cos(x)^2\tan(x) - x^2\sin(x)^2\tan(x) + x^2\sin(x)\cos(x)\sec(x)^2.$$

   (j) Write $h(t) = \sin(t)\sin(t)\sin(t)$ and use the multi-product rule carefully. You should be able to simplify your answer to $h'(t) = 3\sin(t)^2\cos(t)$ using trig identities.

2. We saw that $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$. Therefore, for small values of θ, we have that $\frac{\sin(\theta)}{\theta} \approx 1$. $3°$ is a fairly small angle, so we might want to conclude that $\frac{\sin(3°)}{3} \approx 1$ or equivalently, $\sin(3°) \approx 3$. How can you tell that this is a bad approximation and what went wrong?

Solution. It is a bad approximation because $\sin(\theta)$ is never equal to 3, or very close to 3. What went wrong is when we write $\sin(\theta)$ for some number θ, we always have θ representing an angle in radians, not degrees.

3. Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sin x}{x \cdot 2^x}$

(b) $\lim_{x\to 0} \frac{\tan x}{x}$

(c) $\lim_{\theta\to 0} \frac{\sin(6\theta)}{3\theta}$

(d) $\lim_{y\to 0} \frac{\sin y}{y + \tan y}$

(e) $\lim_{\theta\to 0} \frac{2\theta}{\sin(3\theta)}$

(f) $\lim_{x\to 0} \frac{1 - \cos(x)}{4x\sin(x)}$

(g) $\lim_{t\to 0} \frac{\tan(6t)}{\sin(2t)}$

Solution.

(a) We can split up the limit of products, since each of the two new limits exist.[1]

$$\lim_{x\to 0} \frac{\sin(x)}{x \cdot 2^x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{2^x} = 1 \cdot \frac{1}{2^0} = 1.$$

(b) Write $\frac{\tan(x)}{x} = \frac{\sin(x)}{\cos(x)\cdot x}$. Then

$$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{1}{\cos(x)} = 1 \cdot \frac{1}{\cos(0)} = 1.$$

(c) Let's rewrite $\frac{\sin(6\theta)}{3\theta} = \frac{\sin(6\theta)}{3\theta} \cdot \frac{6\theta}{6\theta} = \frac{\sin(6\theta)}{6\theta} \cdot 2$; multiplying by one so that a matching 6θ term appears. Now because the terms 6θ match, we can use the general fact $\lim_{z\to 0} \frac{\sin(z)}{z} = 1$, because we can set $z = 6\theta$ and $6\theta \to 0$ as $\theta \to 0$. We have:

$$\lim_{\theta\to 0} \frac{\sin(6\theta)}{3\theta} = \lim_{\theta\to 0} \frac{\sin(6\theta)}{6\theta} \cdot 2 = 1 \cdot 2 = 2.$$

(d) The idea is to take the reciprocal of the original limit, which allows us to split up the fraction in a useful way. Writing $\tan(\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$ is also often useful. First:

$$\begin{aligned} \frac{1}{\lim_{y\to 0} \frac{\sin(y)}{y+\tan(y)}} &= \lim_{y\to 0} \frac{1}{\frac{\sin(y)}{y+\tan(y)}} \\ &= \lim_{y\to 0} \frac{y + \tan(y)}{\sin(y)}. \end{aligned}$$

Now this limit is easier to compute:

$$\begin{aligned} \lim_{y\to 0} \frac{y + \tan(y)}{\sin(y)} &= \lim_{y\to 0} \frac{y}{\sin(y)} + \lim_{y\to 0} \frac{\tan(y)}{\sin(y)} \\ &= 1/\left(\lim_{y\to 0} \frac{\sin(y)}{y}\right) + \lim_{y\to 0} \frac{\sin(y)/\cos(y)}{\sin(y)} \\ &= 1/1 + \lim_{y\to 0} \frac{1}{\cos(y)} \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

[1] Remember that if we split up a product limit and obtain a new limit that does not exist, it does not necessarily follow that the original limit does not exist. For example: $\lim_{x\to 0} 1 = (\lim_{x\to 0} x)\left(\lim_{x\to 0} \frac{1}{x}\right)$; we are splitting up the product $1 = x \cdot \frac{1}{x}$. One of the limits on the right-hand side of this equality does not exist, but the limit on the left certainly exists. The same principle holds true with limits of sums.

We figured out:

$$\frac{1}{\lim_{y\to 0}\frac{\sin(y)}{y+\tan(y)}} = 2.$$

Therefore

$$\lim_{y\to 0}\frac{\sin(y)}{y+\tan(y)} = \frac{1}{2}.$$

(e) Similar method to previous one. Do yourself carefully - your answer should be $\frac{2}{3}$.

(f) Your TA found this one quite difficult. A good reminder of the endless joys of mathematics. Think of difficult limit problems like crossword puzzles. You have to *follow the rules* (very important!), and it's very rewarding when you finally crack it.

Multiply by the conjugate is the trick. Then use the most important trig identity.

$$\begin{aligned}\frac{1-\cos(x)}{4x\sin(x)} &= \frac{1-\cos(x)}{4x\sin(x)}\cdot\frac{1+\cos(x)}{1+\cos(x)}\\ &= \frac{1-\cos(x)^2}{(4x\sin(x))(1+\cos(x))}\\ &= \frac{\sin(x)^2}{(4x\sin(x))(1+\cos(x))}\\ &= \frac{\sin(x)}{4x(1+\cos(x))}.\end{aligned}$$

Now we see what to do.

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos(x)}{4x\sin(x)} &= \lim_{x\to 0}\frac{\sin(x)}{4x(1+\cos(x))}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{4(1+\cos(x))}\\ &= 1\cdot\frac{1}{4(1+1)}\\ &= \frac{1}{8}.\end{aligned}$$

(g) We'll multiply by one creatively to add the proper terms to match the limit $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$.

$$\begin{aligned}\frac{\tan(6t)}{\sin(2t)} &= \frac{\sin(6t)}{\cos(6t)}\cdot\frac{1}{\sin(2t)}\\ &= \frac{\sin(6t)}{\cos(6t)}\cdot\frac{6t}{6t}\cdot\frac{1}{\sin(2t)}\cdot\frac{2t}{2t}\\ &= \frac{\sin(6t)}{6t}\cdot\frac{1}{\cos(6t)}\cdot\frac{6t}{1}\cdot\frac{2t}{\sin(2t)}\cdot\frac{1}{2t}\\ &= \frac{\sin(6t)}{6t}\cdot\frac{3}{\cos(6t)}\cdot\frac{2t}{\sin(2t)}\end{aligned}$$

Taking the limit as $t\to 0$, we use the equalities $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$ and $\cos(0)=1$ and find:

$$\begin{aligned}\lim_{t\to 0}\frac{\tan(6t)}{\sin(2t)} &= \lim_{t\to 0}\frac{\sin(6t)}{6t}\cdot\frac{3}{\cos(6t)}\cdot\frac{2t}{\sin(2t)}\\ &= 1\cdot\frac{3}{1}\cdot 1\\ &= 3.\end{aligned}$$

4. Are there any values of k for which the following function is continuous at $x = 0$? If so, find them.

$$f(x) = \begin{cases} \dfrac{x + x\cos x}{\sin x \cos x}, & x \neq 0 \\ k & x = 0 \end{cases}$$

Solution. To ensure $f(x)$ is continuous at $x = 0$, we need to ensure $\lim_{x\to 0} f(x) = k$. So we have to compute $\lim_{x\to 0} f(x)$.

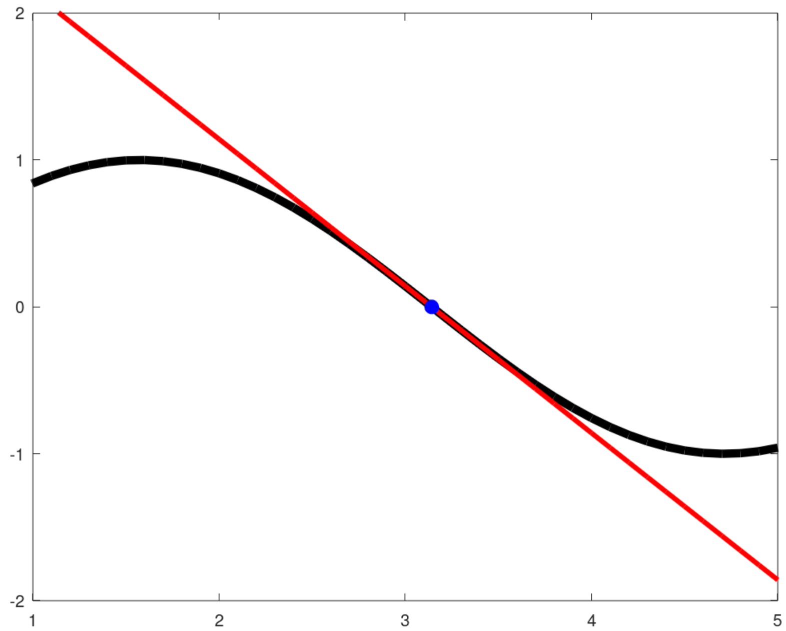
Use the trig identity $2\sin(x)\cos(x) = \sin(2x)$. Then:

$$\begin{aligned} \lim_{x\to 0} \frac{x + x\cos(x)}{\sin(x)\cos(x)} &= \lim_{x\to 0} \frac{x + x\cos(x)}{(1/2)\sin(2x)} \\ &= \lim_{x\to 0} \frac{x}{(1/2)\sin(2x)} + \frac{x\cos(x)}{(1/2)\sin(2x)} \\ &= \lim_{x\to 0} \frac{2x}{\sin(2x)} + \frac{2x}{\sin(2x)} \cdot \cos(x) \\ &= 1 + 1 \cdot 1 \\ &= 2. \end{aligned}$$

5. For $f(x) = \sin(x)$, write an equation for the tangent line to $f(x)$ at $x = \pi$. Sketch a graph of $f(x)$ and this tangent line in the same graph.

Solution. $f'(x) = \cos(x)$, so the slope of the tangent line at $x = \pi$ is $f'(\pi) = \cos(\pi) = -1$. The equation of the tangent line is therefore $T(x) = -(x - \pi) + f(\pi) = -x + \pi$.

Graph:



6. Let $f(x) = \sin^2(x)$ and $g(x) = -\cos^2(x)$. Show that $f'(x) = g'(x)$. Is this surprising? Can you think of another explanation of this?

Solution. I think it is a little surprising. Note that $f(x) = 1 - \cos^2(x)$, according to the most important trig identity. Any two functions that differ by a constant have the same derivative; without doing any computations we can see that it must be that $f'(x) = g'(x)$ since $\frac{d}{dx}(1) = 0$.

Bonus questions - composition of functions.

1. Let $f(t) = e^t$ and $g(t) = \ln(t)$. What is $f(g(t))$? What is $f(g(1))$?

   Solution. $f(g(t)) = e^{\ln(t)} = t$, with the restriction that $t > 0$ since $\ln(t)$ is only defined for $t > 0$. And $f(g(1)) = \ln(1) = 0$.

2. Let $h(q) = 1/q$. What is $h(\csc(\theta))$? What is $h(\csc(\frac{\pi}{4}))$?

   Solution. $h(\csc(\theta)) = \frac{1}{\csc(\theta)} = \frac{1}{1/\sin(\theta)} = \sin(\theta)$, with the restriction that θ is not a multiple of π, since $\csc(\theta)$ is not defined when $\sin(\theta) = 0$, which happens when θ is a multiple of π. We have $h(\csc(\frac{\pi}{4})) = \sin(\frac{\pi}{4}) = 2^{-1/2}$.

3. Consider the function $f(x) = \sqrt{1 + x^2}$. Find functions $g(x)$ and $h(x)$ such that $f(x) = g(h(x))$.

   Solution. $g(x) = \sqrt{x}$ and $h(x) = 1 + x^2$.