1. Each of the following functions can be written as a composition. Find $f(x)$ and $g(x)$ so that the following function is of the form $f(g(x))$. Then, find the derivative of the function using the chain rule.

   (a) $y = (2x+1)^2$. Can you take the derivative another way to check your work? How many ways can you take this derivative?

   (b) $y = \sin(4x)$

   (c) $y = \sqrt{2+x^2} + (2+x^2)^3$

   (d) $y(z) = \sqrt{\dfrac{z-1}{z+1}}$

Solution.

(a) Let $f(x) = x^2$ and $g(x) = 2x+1$. Then $f(g(x)) = (2x+1)^2 = y(x)$. Using the chain rule and the formulas $f'(x) = 2x$ and $g'(x) = 2$, we have:

$$y'(x) = (f(g(x))' = f'(g(x))g'(x) = 2(2x+1)\cdot 2 = 8x+4.$$

(b) Let $f(x) = \sin(x)$ and $g(x) = 4x$. Then $f(g(x)) = \sin(4x) = y(x)$. Using the chain rule and the formulas $f'(x) = \cos(x)$ and $g'(x) = 4$, we have:

$$y'(x) = (f(g(x))' = f'(g(x))g'(x) = \cos(4x)\cdot 4 = 4\cos(4x).$$

(c) Split up the function as a sum when taking the derivative.

$$y'(x) = \frac{d}{dx}\left(\sqrt{1+x^2}\right) + \frac{d}{dx}\left((2+x^2)^3\right).$$

For the first function, take $f(x) = \sqrt{x} = x^{1/2}$ and $g(x) = 1+x^2$; then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$ and $g'(x) = 2x$. For the second function, take $f(x) = x^3$ and $g(x) = 2+x^2$. Then $f'(x) = 3x^2$ and $g'(x) = 2x$. Then, from the chain rule on each derivative:

$$\begin{aligned} y'(x) &= \frac{1}{2}(1+x^2)^{-1/2}(2x) + 3(2+x^2)^2(2x) \\ &= x(1+x^2)^{-1/2} + 6x(2+x^2)^2. \end{aligned}$$

(d) Let $f(z) = \sqrt{z} = z^{1/2}$ and $g(z) = \frac{z-1}{z+1}$. Then $f'(z) = \frac{1}{2}z^{-\frac{1}{2}}$ and $g'(z) = \frac{(z+1)-(z-1)}{(z+1)^2} = \frac{2}{(z+1)^2}$. From the chain rule:

$$\begin{aligned} y'(z) &= f'(g(z))g'(z) \\ &= \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}} \cdot \frac{2}{(z+1)^2} \\ &= \left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} \cdot \frac{1}{(z+1)^2} \end{aligned}$$

2. A differentiable function f satisfies $f(3) = 5, f(\frac{1}{9}) = 7, f'(3) = 11$, and $f'(\frac{1}{9}) = 13$. Let $g(x) = f(\frac{1}{x^2})$. Find the equation to the tangent line of $y = g(x)$ when $x = 3$.

Solution. The tangent line passes through the point $(3, g(3)) = (3, f(\frac{1}{9})) = (3, 7)$. Using the chain rule, noting that $\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx}x^{-2} = -2x^{-3}$, we find the derivative

$$y'(x) = g'(x) = f'(\frac{1}{x^2}) \cdot (-2x^{-3}).$$

The slope of the tangent line is then:

$$y'(3) = f'(\frac{1}{9}) \cdot (-2) \cdot (\frac{1}{9})^{-3} = 13 \cdot (-2) \cdot 9^3 = -18954.$$

And therefore the equation of the tangent line is

$$T(x) = -18954(x - 3) + 7.$$

3. Suppose $f(x)$ and $g(x)$ are functions whose values and derivatives at $x = 0$ and $x = 1$ are given in the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	-4	$-1/3$	$-8/3$

We define the following functions:

- $v(x) = f(g(x))$
- $w(x) = g(f(x))$
- $p(x) = f(x)g(x)$
- $q(x) = \frac{g(x)}{f(x)}$

Determine $v(0), w(0), p(0), q(0)$ and $v'(0), w'(0), p'(0), q'(0)$. If there isn't enough information to compute a value, state so.

Solution. We use the chain rule for the first two, product rule for the third one, and the quotient rule for the last one.

- $v'(0) = f'(g(0))g'(0) = f'(1)(1/3) = 3 \cdot (1/3) = 1.$
- $w'(0) = g'(f(0))f'(0) = g'(1) \cdot 5 = (-8/3) \cdot 5 = -40/3.$
- $p'(0) = f'(0)g(0) + f(0)g'(0) = 5 \cdot 1 + 1 \cdot (-1/3) = 5 - 1/3 = 14/3.$
- $q'(0) = \frac{f(0)g'(0) - g(0)f'(0)}{f(0)^2} = \frac{1 \cdot (1/3) - 1 \cdot 5}{1^2} = -14/3.$

4. Compute the derivative of $y = \tan(\sqrt{1 + x^2})$.

Solution. Chain rule (2×).

$$y'(x) = \sec^2(\sqrt{1+x^2}) \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot (2x) = \frac{x}{\cos^2(\sqrt{1+x^2})\sqrt{1+x^2}}.$$

5. Compute the derivative of $h(x) = \left(\dfrac{\sec(x)}{1+x^2}\right)^{1/3}$.

Solution. Chain rule.

$$h(x) = \frac{1}{3}\left(\frac{\sec(x)}{1+x^2}\right)^{-2/3}\left(\frac{(1+x^2)\sec(x)\tan(x) - 2x\sec(x)}{(1+x^2)^2}\right).$$

6. Compute the derivative of $y = \tan((\sin(2x))^2)$.

Solution. Chain rule.

$$y'(x) = \sec^2(\sin(2x)^2) \cdot 2\sin(2x) \cdot \cos(2x) \cdot 2.$$

7. Find $\dfrac{d^2y}{dx^2}$ if $y = (3x + \tan(x))^3$.

***Note:* On exams, especially for complex problems such as this, you must write your algebra in a neat and organized manner in order to receive credit!!**

Solution. First, we have by the chain rule,

$$\frac{dy}{dx} = 3(3x + \tan(x))^2(3 + \sec^2(x)).$$

Then by the product rule and chain rule,

$$\begin{aligned}\frac{d^2y}{dx^2} &= 3 \cdot 2(3x+\tan(x))(3+\sec^2(x))(3+\sec^2(x)) + 3(3x+\tan(x))^2(2\sec(x)\cdot\sec(x)\tan(x)) \\ &= 6(3x+\tan(x))(3+\sec^2(x))^2 + 6(3x+\tan(x))^2\sec^2(x)\tan(x)\end{aligned}$$

8. Consider the curve defined by $x^2 + y^2 = 1$.

(a) Is y a function of x? How can you see this graphically? How can you see this algebraically?

Solution. No, the unit circle does not pass the "vertical line test." For example, the line

$$x = 0$$

passes through *two* points on the circle, $(0, 1)$ and $(0, -1)$, not one. Algebraically, solving for y we have

$$\begin{aligned} y^2 &= 1 - x^2 \\ \Rightarrow y &= \pm\sqrt{1 - x^2}. \end{aligned}$$

Each x-coordinate determines two possible values of y, not 1.

(b) Solve for y as a function of x on the bottom half of the circle. Find y'.

Solution. Solving $y^2 = 1 - x^2$ for the "top half," where $y \geq 0$, we have $y = \sqrt{1 - x^2}$. Now we may use the chain rule:

$$y' = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}}.$$

(c) Solve for y as a function of x on the top half of the circle. Find y'.

Solution. Solving $y^2 = 1 - x^2$ with $y \leq 0$ gives us $y = -\sqrt{1 - x^2}$. The derivative is now:

$$y' = \frac{x}{\sqrt{1 - x^2}}.$$

(d) Use implicit differentiation to find y'. How does this answer relate to your answer from the previous parts?

Solution. We start with $x^2 + y^2 = 1$, viewing $y = y(x)$.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(1) \\ \Rightarrow 2x + 2yy' &= 0 \\ \Rightarrow 2yy' &= -2x \\ \Rightarrow y' &= \frac{-2x}{2y} = \frac{-x}{y}. \end{aligned}$$

In order to find y' as a function of x, we have to find y as a function of x. Like before, we can either choose $y = \sqrt{1 - x^2}$ or $y = -\sqrt{1 - x^2}$. Plugging the second formula for y, for example, we find y' from part (c):

$$y' = \frac{-x}{y} = \frac{-x}{-\sqrt{1 - x^2}} = \frac{x}{\sqrt{1 - x^2}}.$$

9. Use implicit differentiation to find $\frac{dy}{dx}$ if $y^2 + xy = 4x^2$.

Solution. Chain rule and product rule. Remember we are assuming y is a function of x: $y = y(x)$.

$$\begin{aligned} \frac{d}{dx}\left(y^2 + xy\right) &= \frac{d}{dx}\left(4x^2\right) \\ 2yy' + y + xy' &= 8x \\ (2y + x)y' &= 8x - y \\ y' &= \frac{8x - y}{2y + x}. \end{aligned}$$

We have found y' as a function of (x, y).

10. Use implicit differentiation to find $\frac{dy}{dx}$ if $\cos\left(\frac{x}{y}\right) = x\sin(y)$.

Solution. Take the derivative of both sides of the equation $\cos\left(\frac{x}{y}\right) = x\sin(y)$ with respect to x. Then solve for y'.

$$\begin{aligned}
-\sin\left(\frac{x}{y}\right)\cdot\frac{y-xy'}{y^2} &= \sin(y) + x\cos(y)y' \\
\Rightarrow -\sin\left(\frac{x}{y}\right)\cdot\frac{y-xy'}{y^2} &= \sin(y) + x\cos(y)y' \\
\Rightarrow -\sin\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y} - \frac{x}{y^2}y'\right) &= \sin(y) + x\cos(y)y' \\
\Rightarrow \frac{-1}{y}\sin\left(\frac{x}{y}\right) + \frac{x}{y^2}\sin\left(\frac{x}{y}\right)y' &= \sin(y) + x\cos(y)y' \\
\Rightarrow \left(\frac{x}{y^2}\sin\left(\frac{x}{y}\right) - x\cos(y)\right)y' &= \sin(y) + \frac{1}{y}\sin\left(\frac{x}{y}\right) \\
\Rightarrow y' &= \frac{\sin(y) + \frac{1}{y}\sin\left(\frac{x}{y}\right)}{\left(\frac{x}{y^2}\sin\left(\frac{x}{y}\right) - x\cos(y)\right)}.
\end{aligned}$$

11. Find the equation of the tangent line to the curve defined by $\cos(x^2y) = 3xy^2 + y$ at the point $(0, 1)$.

Solution. First, we use implicit differentiation to find dy/dx as a function of (x, y).

$$\begin{aligned}
-\sin(x^2y)(2xy + x^2y') &= 3y^2 + 6xyy' + y' \\
\Rightarrow -2xy\sin(x^2y) - x^2\sin(x^2y)y' &= 3y^2 + (6xy+1)y' \\
\Rightarrow -2xy\sin(x^2y) - 3y^2 &= (x^2\sin(x^2y) + 6xy + 1)y' \\
\Rightarrow y' &= \frac{-2xy\sin(x^2y) - 3y^2}{x^2\sin(x^2y) + 6xy + 1}
\end{aligned}$$

Now plug in $(x, y) = (0, 1)$.

$$y'(0,1) = \frac{-2(0)(1)\sin((0)^2(1)) - 3(1)^2}{(0)^2\cdot 0 + 6(0)(1) + 1} = -3.$$

The tangent line has slope $y'(0, 1) = -3$ and passes through the point $(0, 1)$. The equation of this line is

$$T(x) = -3(x-0) + 1 = -3x + 1.$$

12. Find a point on the curve $x^2 = y^3 + y + 2$ and use implicit differentiation to find the equation of the tangent line through that point.

Solution. Let's try setting $y = 0$ and see if we can solve for x. We obtain $x^2 = 2$, and $x = \sqrt{2}$ is a solution of this equation. Therefore $(x, y) = (\sqrt{2}, 0)$ is a point on the curve.

Implicit derivative:

$$2x = 3y^2y' + y' \qquad \Rightarrow \qquad y' = \frac{2x}{3y^2+1}.$$

Inserting $(x, y) = (\sqrt{2}, 0)$, we have:

$$y' = \frac{2\sqrt{2}}{3(0)^2+1} = 2\sqrt{2}.$$

Using the point-slope equation of a line, we find the tangent line is:

$$y = 2\sqrt{2}(x - \sqrt{2}) + 0.$$