Math 221 - Week 9 - Worksheet 2
Topics: Section 4.2 - The Definite Integral

Instructions: Listen to your TA's instructions. There are substantially more problems on this worksheet than we expect to be done in discussion, and your TA might not have you do problems in order. The worksheets are intentionally longer than will be covered in discussion in order to give students additional practice problems they may use to study. Do not worry if you do not finish the worksheet :).

1. Let $f(x) = x$

   (a) Sketch $f(x)$ on $[-3, 3]$

   (b) What is the area bounded by $f(x)$ and the x-axis on this interval?

   (c) What is $\int_{-3}^{3} f(x)dx$?

   (d) Why are parts (b) and (c) not the same?

2. Let $f(x) = 1 + \sqrt{9 - x^2}$.

   (a) Sketch $f(x)$ on the interval $[-3, 0]$. Use your picture to calculate the area under the curve on this interval.

(b) What is $\int_{-3}^{0} 1+\sqrt{9-x^2}dx$?

(c) Use the same technique to calculate $\int_{-3}^{0} \sqrt{9-x^2}dx$ and $\int_{-3}^{0} 1dx$ (sketch the functions $g(x) = \sqrt{9-x^2}$ and $h(x) = 1$ on the interval $[-3, 0]$).

(d) Is it true that $\int_{-3}^{0} 1+\sqrt{9-x^2}dx = \int_{-3}^{0} 1dx + \int_{-3}^{0} \sqrt{9-x^2}dx$?

3. Let $f(x) = 2x$, $g(x) = x+1$.

(a) Sketch $f(x)$ and $g(x)$ on the interval $[0, 2]$. Use areas to calculate $\int_{0}^{2} f(x)dx$ and $\int_{0}^{2} g(x)dx$.

(b) Sketch $f(x)+g(x)$ on the interval $[0, 2]$. Use areas to calculate $\int_{0}^{2} \Big(f(x)+g(x)\Big)dx$.

Is it equal to $\int_{0}^{2} f(x)dx + \int_{0}^{2} g(x)dx$?

(c) Sketch $f(x)g(x)$ on the interval $[0, 2]$.

Use your picture to guess if $\left(\int_0^2 f(x)dx\right)\left(\int_0^2 g(x)dx\right) = \int_0^2 f(x)g(x)dx$.

(d) Sketch $\frac{1}{2}f(x)$ on the interval $[0, 2]$. Use areas to find $\int_0^2 \left(\frac{1}{2}f(x)\right)dx$.

Is it equal to $\frac{1}{2}\int_0^2 f(x)dx$?

4. Compute the following.

(a) $\int_{-2}^{5} 3x + 2dx$

(b) $\int_{-1}^{1} \sqrt{1-x^2}dx$.

(c) $\int_{-1}^{2} |x|dx$

5. Suppose f, g are continuous functions on $[0,4]$ where $\int_0^1 f(x)dx = 4$, $\int_0^4 f(x)dx = -6$, $\int_0^1 g(x)dx = -2$, and $\int_1^4 g(x)dx = 13$.

Calculate the following.

(a) $\int_1^4 f(x)dx + \int_1^1 g(x)dx$

(b) $\int_0^4 f(x) - g(x)dx$

(c) $\int_4^1 2f(x) + 3g(x)dx$

6. As we saw in the previous problem, the integral has a lot of nice properties. We can pull out constants and split it up over addition. Can we pull out squares? In this problem, we will compare $\int_a^b f(x)^2 dx$ and $\left(\int_a^b f(x)dx\right)^2$. For simplicity, let's use the function $f(x) = x$.

(a) Use geometry to calculate $\int_0^3 x dx$. Call this value A.

(b) Use a right hand sum with 3 rectangles to determine an estimate for $\int_0^3 x^2 dx$. Call this estimate B. Draw a sketch of the graph of $y = x^2$ and the rectangles whose area you computed.

(c) How do A^2 and B compare? Do you think $\int_0^3 x^2 dx = \left(\int_0^3 x dx\right)^2$?

(d) In general, an estimate for $\int_0^3 x^2 dx$ is not enough to determine whether $\int_0^3 x^2 dx$ is equal to $\left(\int_0^3 x dx\right)^2$. However, we can be clever in this case. Is your estimate B an overestimate or an underestimate for the actual value of the integral? Use this to determine if $\int_0^3 x^2 dx$ is equal to $\left(\int_0^3 x dx\right)^2$.

7. True or False? Justify your answer. $2 \leq \int_{-1}^{1} \sqrt{1+x^2}\, dx \leq 2\sqrt{2}$.

8. Let $f(x) = 2x$.

(a) Sketch $f(x)$, and label a positive point z on the x-axis.

(b) Calculate the area under the curve on the interval $[0, z]$. Write down a function $F(z)$ which represents the area.

(c) Use areas to calculate $\int_{-2}^{1} f(x)dx$.

(d) How does $F(x)$ relate to $f(x)$?

(e) Calculate $F(1)-F(-2)$, and compare this to the integral you calculated. Use this to guess a rule for calculating integrals without using areas.