Introduction to Boij-Söderberg Theory

2020-11-11 John Cobb

Outline of Talk

1. Types of free resolutions
2. Betti Tables and some geometry
3. Boij-Söderberg Conjectures & Interpretations
4. Connection to cohomology of vector bundles on projective spaces

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Notation:
- $A = k[x_1, \dots, x_n]$, k a field.
- M finitely generated A-module.

High Level View: There is a "duality" between two numerical invariants:

"Most" General

{ Betti tables of graded free resolutions over $k[x_1, \dots, x_n]$ } $\longleftrightarrow$ { cohomology tables of coherent sheaves on projective spaces }

Boij-Söderberg (2008)

Eisenbud Schreyer (2009)

Original

{ Betti tables of graded free resolutions of cohen Macaulay modules } $\overset{\text{Eisenbud}}{\underset{\text{Schreyer (2007)}}{\longleftrightarrow}}$ { cohomology table of vector bundles on projective spaces }

Boij-Söderberg (2006)

Question: How can we describe the structure of a module?

Definition. An exact sequence of A-modules

$$\mathbb{F}: \cdots \xrightarrow{d_{i+2}} F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

is a free resolution of M if F_i is free $\forall i$.

Note: Every module has at least one of these.

Construction:

$$\begin{tikzcd} F_2 \arrow[r] \arrow[dr] & F_1 \arrow[r, dashed] \arrow[dr] & F_0 \arrow[r, "d_0"] & M \arrow[r] & 0 \\ & \ker d_1 \arrow[u] & \ker d_0 \arrow[u] & & \end{tikzcd}$$

0^{th} syzygy of M.

Ex// Let $A = k[x,y,z]$ and $M = (xy, xz)$

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} -y \\ 1 \end{pmatrix}} A^2 \xrightarrow{\begin{pmatrix} z & yz \\ -y & -y^2 \end{pmatrix}} A^2 \xrightarrow{(xy \;\; xz)} M \longrightarrow 0$$

$$-y\begin{pmatrix} z \\ -y \end{pmatrix} + 1\begin{pmatrix} yz \\ -y^2 \end{pmatrix} = 0. \quad z(xy) - y(xz) = 0.$$

$$\begin{pmatrix} yz \\ -y^2 \end{pmatrix} = y\begin{pmatrix} z \\ -y \end{pmatrix}$$

$$0 \longrightarrow A \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} A^2 \xrightarrow{(xy \;\; xz)} M \longrightarrow 0$$

$$0 \longrightarrow A(-3) \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} A(-2) \oplus A(-2) \xrightarrow{(xy \;\; xz)} M \longrightarrow 0$$

f degree $n-3$ $\qquad$ f degree $n-2$ $\qquad$ f deg n.

We want to construct these resolutions as efficiently as possible
$\rightarrow$ At each step in our construction, we want a "minimal" set of relations

✓ Our concept of minimality will only make sense (i.e. yield uniqueness) when

- modules over local rings (R, m)
- graded modules over graded rings $(A, m = (x_1, \ldots, x_n))$

} Nakayama's lemma.

$$A = \bigoplus_{i=0}^{n} A_i \qquad A_i = i \text{ degree monomials.}$$

$$A_i \cdot A_j \subseteq A_{i+j}.$$
$$A_i \cdot M_j \subseteq M_{i+j}$$

$$M(-r) = M_{j-r}. \qquad \text{e.g.) } A(-1)_1 = A(0) = k.$$

Definition. A free resolution is **minimal** if $d_{i+1}(F_{i+1}) \subseteq \mathfrak{m} F_i \ \forall i \geq 0$

Idea: This is a slick way of saying that we are choosing the minimal # of generators of our free modules at each step of our construction.

Nakayama $\Longrightarrow$ basis of $M/\mathfrak{m}M$ (a $A/\mathfrak{m} = k$-vector space) lifts to a minimal generating set of M as an A-module.

$$\begin{tikzcd}
0 \arrow[r] & \ker d_0 \arrow[r] \arrow[d] & F_0 \arrow[r, "d_0"] \arrow[d, "\otimes_A A/\mathfrak{m}"] & M \arrow[r] & 0 \\
& 0 & F_0/\mathfrak{m}F_0 \arrow[r, "\sim"] & M/\mathfrak{m}M \arrow[r] & 0
\end{tikzcd}$$

$$\ker d_0 / \mathfrak{m} F_0 = 0 \implies \underset{\operatorname{im} d_1}{\overset{}{\ker d_0}} \subseteq \mathfrak{m} F_0$$

Question: Does this process of making a minimal free resolution terminate? Why do we care?

Hilberts Syzygy Theorem. (1890) The n^{th} syzygy is a free module!

$(\Longrightarrow)$ M always has a unique graded free resolution whose length (which is called the **projective dimension** of M or $\operatorname{pd}(M)$) is bounded by $\dim A = n$.

In this situation, $F_i = \bigoplus_j A(-j)^{\beta_{i,j}}$ — Betti numbers of M

Ex// Let $A = k[x,y,z]$ and $M = (xy, xz)$

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} \underbrace{A(-2) \oplus A(-2)}_{1} \xrightarrow{(xy\ xz)} M \to 0$$

Betti numbers $\beta_{i,j}$ can be written into a **Betti table**

length

$$(\beta_{ij}) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & \cancel{0} \\ 3 & 0 & 1 \end{array}$$

degree shifts

degree goes down by at least 1 so shift the j^{th} col up by j.

$$\rightsquigarrow \quad \beta(M) = (\beta_{i,i+j}) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 1 \end{array}$$

Ex//

$$0 \longrightarrow A(-4) \oplus A(-3) \xrightarrow{\begin{pmatrix} 0 & x \\ x & -y^2 \\ -y & 0 \end{pmatrix}} A(-3) \oplus A(-2)^2 \xrightarrow{(y^3\ xy\ x^2)} A \longrightarrow M \longrightarrow 0$$

$$A/(y^3\ xy\ x^2) = M$$

$$(\beta_{ij}) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & \times & \times \\ 2 & & 2 & \times \\ 3 & & 1 & 1 \\ 4 & & & 1 \end{array} \quad \rightsquigarrow \quad \beta(M) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 2 & 1 \\ 2 & & 1 & 1 \end{array}$$

General Direction of Research: How do the properties of the minimal free resolution + its numerical invariants (e.g. Betti #'s) relate to the structure of M or of A?

Our Direction: Understand the possible Betti tables that can occur.

Ex//

	0	1	2	3
0	1	2	0	
1			2	1

is not the Betti table of <u>any</u> module.

Why? 2 linear relations, x and y,

$$y(x) - x(y) = 0$$

	0	1	2	3
0	2	4		
1			2	4

is the betti table of the minimal free resolution

$$0 \longrightarrow \underline{A(-4)^2} \xrightarrow{\begin{pmatrix} x_{14} & x_{24} \\ -x_{13} & -x_{23} \\ x_{12} & x_{22} \\ -x_{11} & -x_{21} \end{pmatrix}} \underline{A(-3)^4} \xrightarrow{\begin{pmatrix} \Delta_{23} & \Delta_{24} & \Delta_{34} & 0 \\ -\Delta_{13} & -\Delta_{14} & 0 & \Delta_{34} \\ \Delta_{12} & 0 & -\Delta_{14} & -\Delta_{24} \\ 0 & \Delta_{12} & \Delta_{13} & \Delta_{23} \end{pmatrix}} A(-1)^4 \xrightarrow[\substack{\Delta_{ij} = 2\times 2 \text{ minor} \\ \text{at } (i,j)}]{\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & 2x_2 & 3x_3 & 4x_4 \end{pmatrix}} A^2$$

$\beta_{ij} =$

	0	1	2	3	4
0	2	X			
1		4	X	X	
2			X	X	
3			4	X	
4				2	

pure!

$d = (0, 1, 3, 4)$

Definition: $\mathbb{F}$ is pure if each free module F_i is generated in a single degree. $F_i = A(-d_i)^{\beta_{i,d_i}}$.

- $\mathbb{F}$ has degree sequence $d = (d_0 < d_1 < \cdots < d_n)$
- If $\mathbb{F}$ is pure, we denote its Betti table as $\pi(M)$.

Key Insight of Boij–Söderberg:

Classifying <u>all</u> Betti tables is hard. (its still completely out of reach!)

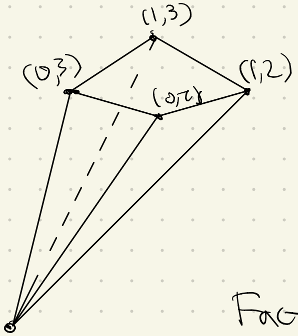
Our New Direction: Understand the possible Betti tables that can occur.*

* up to rational multiples.

Think of a Betti table $\beta(M) = (\beta_{ij}(M))$ as a vector in $\underbrace{\bigoplus_{j \in \mathbb{Z}}}_{\text{arbitrary relation}} \mathbb{Q}^{n+1}$ ← columns bounded by n

Fact. The set of positive rational rays $q \cdot \beta(M)$ $(q \in \mathbb{Q}_{\geq 0})$ forms a $\underbrace{\text{convex}}_{\alpha x + \beta y \in C}$ $\underbrace{\text{cone}}_{x \in C \Rightarrow \alpha x \in C}$.

Pf. (Sketch) If M + N are modules, + $m, n \in \mathbb{Q}$. then $[M^m \oplus N^n]$ has the Betti table $m\beta(M) + n\beta(N)$.



Turns out!
the extremal rays of this cone are exactly the pure Betti tables

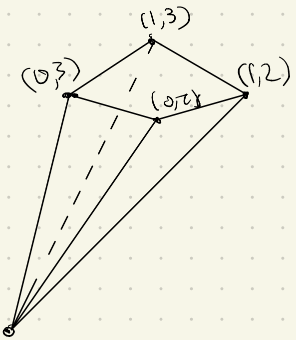
Fact: Betti table can be realized as a positive rational sum of pure Betti tables
$$\beta(M) = r_1 \pi(N_1) + \cdots + r_n \pi(N_n) \qquad r_i \in \mathbb{Q}_{\geq 0}$$

Boij–Söderberg Conjectures (Stated 2006, Solved by Eisenbud + Schreyer 2007)

1. Existence. For every degree sequence $d = (d_0 < d_1 < \cdots < d_n)$ there exists a Cohen–Macaulay module with a pure resolution of type d.

2. Spanning. The cone of Betti tables is generated by pure Betti tables

3. Decomposition. Every Betti table is a unique positive rational linear combination of pure Betti tables in a unique chain of degree sequences.



a

Wait, where did Cohen-Macaulayness come from?

Needed to say that degree sequences unambiguously define a pure resolution up to rational multiples.

But if you are willing to let the pure Betti tables in

$$\beta(M) = r_1 \pi(N_1) + \cdots + r_n \pi(N_m)$$

have different number of columns, we can throw Cohen-Macaulay out.

eg) $M = A/(x^2, xy, xz, yz)$ not Cohen-Macaulay.

$$\beta(M) = \begin{bmatrix} 1 & & & \\ & 4 & 4 & 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & & & \\ & 6 & 8 & 3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 & & \\ & 3 & 2 \end{bmatrix}$$

But what is it?

There is this other notion of dimension of a module called depth.
We know depth $M \leq \dim M$. Equality iff M Cohen-Macaulay

Theorem (Auslander-Buchsbaum) $pd(M) = \text{depth}(A) - \text{depth}(M) \rightsquigarrow pd(M) = n - \dim(M)$

Why is this relevant? If we now define Hilbert series and Hilbert functions, we can use this fact to get some relations that our betti #'s need to satisfy:

$$\begin{bmatrix} 1 & -1 & \cdots & (-1)^c \\ d_0 & -d_1 & \cdots & (-1)^c d_c \\ \vdots & & & \\ d_0^{c-1} & -d_1^{c-1} & \cdots & (-1)^c d_c^{c-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \dim M \end{bmatrix} = 0.$$

(Herzog-Kühl equations)

Ex//

$$A/(y^3, xy, x^2) = M$$

$$0 \longrightarrow A(-4) \oplus A(-3) \xrightarrow{\begin{pmatrix} 0 & x \\ x & -y^2 \\ -y & 0 \end{pmatrix}} A(-3) \oplus A(-2)^2 \xrightarrow{(y^3 \ xy \ x^2)} A \longrightarrow M \longrightarrow 0$$

$$\beta(M) = (\beta_{i, j+i}) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 2 & 1 \\ 2 & & 1 & 1 \end{array}$$

$$\pi(0,2,3) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 3 & 2 \\ 2 & & & \end{array} \qquad \pi(0,2,4) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 2 & \\ 2 & & & 1 \end{array} \qquad \pi(0,3,4) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & & \\ 2 & & 4 & 3 \end{array}$$

$$\beta(M) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & & \\ 1 & & 2 & 1 \\ 2 & & 1 & 1 \end{array} = \frac{1}{2}\pi(0,2,3) + \frac{1}{4}\pi(0,2,4) + \frac{1}{4}\pi(0,3,4)$$

BS decomposition.

CONNECTION TO VECTOR BUNDLES

To prove

③ Decomposition. Every Betti table is a unique positive rational linear combination of pure Betti tables in a unique chain of degree sequences.

Eisenbud & Schreyer discovered a surprising connection to vector bundles on projective spaces + their "cohomology tables."

High Level View

Studied facets of the positive cone of Betti tables. They found that the coefficients of supporting hypersurfaces were cohomological dimensions of a certain class of vector bundles on projective spaces.

☆ Need to assume some familiarity with sheaves + homological algebra ☆

Definition. Over $\mathbb{P}^n$, the sheaf $\mathcal{E}$ is quasi-coherent if every affine subscheme $U = \operatorname{Spec} A$, $\mathcal{E}|_U$ is the sheaf associated to the module $M = \Gamma(U, \mathcal{E})$ over A.

$\uparrow$ $\mathcal{E}(U)$

- Over $\mathbb{P}^n$, $\mathcal{E}$ is coherent if it is quasi-coherent and $\Gamma(U,\mathcal{E})$ are all finitely generated (as a module)

Sheaf Cohomology

Fix a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$. The global sections functor

$$\Gamma : \text{Sheaves}(\mathbb{P}^n) \longrightarrow Ab$$
$$\mathcal{F} \longmapsto \mathcal{F}(X)$$

is right adjoint to the constant sheaves functor, so Γ is left exact.

(So it makes sense to talk about the right derived functors of Γ)

$$H^i(\mathbb{P}^n, \mathcal{E}) = R^i\Gamma(\mathcal{E})$$

$0 \to A \to B \to C \to 0$

$\rightsquigarrow$

is the sheaf cohomology of $\mathcal{E}$ on $\mathbb{P}^n$.

$0 \to \Gamma(A) \to \Gamma(B) \to \Gamma(C)$

$\downarrow$

$R^1\Gamma(\mathcal{E})$

Definition: The cohomology table of $\mathcal{E}$ is given by

$$\gamma(\mathcal{E}) = \gamma_{i,j}(\mathcal{E}) = \dim_k H^i(\mathbb{P}^n, \mathcal{E}(j))$$

$\hookrightarrow \mathcal{E} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(j)$

Ex//.

$$\gamma(\mathcal{O}_{\mathbb{P}^2}) = \underbrace{\begin{array}{ccccccccc} \cdots & - & 3 & 1 & - & - & - & - & \cdots \\ \cdots & - & - & - & - & - & - & - & \cdots \\ \cdots & - & - & - & 1 & \boxed{3} & 6 & 10 & \cdots \\ \hline \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \end{array}}_{\text{shifts (twists)}} \Bigg] H^i \qquad \mathcal{O}_{\mathbb{P}^n}(j)$$

global section is exactly the j^{th} degree homogeneous polynomials.

$$\gamma_{0,1}(\mathcal{O}_{\mathbb{P}^2}(1)) = \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$$

supernatural

$$= \dim \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$$

$$= \dim (\mathcal{O}_{\mathbb{P}^2}(1))(\mathbb{P}^2) \qquad k[x_0, x_1, x_2]$$

$$= \dim (x_0, x_1, x_2) = 3.$$

To find all betti tables up to rational multiple, we needed the extremal rays

Definition. A vector bundle on a projective space is a supernatural vector bundle if each column has exactly <u>one</u> nonzero entry!

Where did $\underbrace{\text{vector}}$ bundles come from?

locally free coherent sheaves.

$$x \in \mathbb{P}^n \; \exists\, U \ni x \text{ s.t. } \mathcal{E}|_U = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}|_U.$$

Before, the degree sequences gave us a cohen macaulay module & the association was unique up to rational multiple.

Fact. A vector bundle is supernatural iff $\exists$ a sequence $r_m < r_{m-1} < \cdots < r_1$ (root sequence) satisfying analagous properties to degree sequences

Given a free resolution $\mathbb{F}$ (length $\leq n+1$) + coherent sheaf $\mathcal{E}$ on $\mathbb{P}^n$
Eisenbud & Schreyer defined a pairing

$$\langle \mathbb{F}, \mathcal{E} \rangle_{e,t} \quad (e \in \mathbb{Z}, t = 0, \ldots, n)$$

defined in terms of Betti tables of $\mathbb{F}$ + cohomology tables of $\mathcal{E}$.

Flavor:

- The supporting hyperplanes for the Betti tables of Cohen-Macaulay modules of codim c are given by $\langle -, \mathcal{E} \rangle_{e,t}$.
- The supporting hyperplanes of the cone of cohomology tables of vector bundles on $\mathbb{P}^{n-c}$ is given by $\langle \mathbb{F}, - \rangle_{e,t}$.

$$\Downarrow$$

Thm. For any vector bundle $\mathcal{E}$ there is a unique chain of root sequences $r^1 < \cdots < r^p$ s.t. the cohomology tables of $\mathcal{E}$ is

$$c_1 \gamma(r^1) + c_2 \gamma(r^2) + \cdots + c_p \gamma(r^p)$$

c_i are positive and rational.

From here:

- Generalize to rings other than $k[x_1, \ldots, x_n]$ (Erman), Toric Varieties
- Still need to fully generalize from vector bundles to coherent sheaves (we haven't fully classified those cohomology tables)
- Related conjectures: Horrocks, Stillmans, minimal model program.

References:

QUESTIONS ABOUT BOIJ–SÖDERBERG THEORY

DANIEL ERMAN AND STEVEN V SAM

1. Background on Boij–Söderberg Theory

Boij–Söderberg theory focuses on the properties and duality relationship between two types of numerical invariants. One side involves the Betti table of a graded free resolution over the polynomial ring. The other side involves the cohomology table of a coherent sheaf on projective space. The theory began with a conjectural description of the cone of Betti tables of finite length modules, given in [10]. Those conjectures were proven in [25], which also described the cone of cohomology tables of vector bundles and illustrated a sort of duality between Betti tables and cohomology tables.

The theory itself has since expanded in many directions: allowing modules whose support has higher dimension, replacing vector bundles by coherent sheaves, working over rings other than the polynomial ring, and so on. But at its core, Boij–Söderberg theory involves:

(1) A classification, up to scalar multiple, of the possible Betti tables of some class of objects (for example, free resolutions of finitely generated modules of dimension $\leq c$).
(2) A classification, up to scalar multiple, of the cohomology tables of some class of

BOIJ-SÖDERBERG THEORY: INTRODUCTION AND SURVEY

GUNNAR FLØYSTAD

Abstract. Boij-Söderberg theory describes the Betti diagrams of graded modules over the polynomial ring, up to multiplication by a rational number. Analog Eisenbud-Schreyer theory describes the cohomology tables of vector bundles on projective spaces up to rational multiple. We give an introduction and survey of these newly developed areas.

Contents

March 2016

THREE THEMES OF SYZYGIES

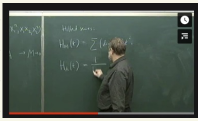
GUNNAR FLØYSTAD, JASON MCCULLOUGH, AND IRENA PEEVA

Abstract. We present three exciting themes of syzygies, where major progress was made recently: Boij-Söderberg Theory, Stillman's Question, and Syzygies over Complete Intersections.

Free Resolutions are both central objects and fruitful tools in Commutative Algebra. They have many applications in Algebraic Geometry, Computational Algebra, Invariant Theory, Hyperplane Arrangements, Mathematical Physics, Number Theory, and other fields. We introduce and motivate free resolutions and their invariants in Sections 1 and 3. The other sections focus on three hot topics, where major progress was made recently:

- Syzygies over Complete Intersections (see Section 2),
- Stillman's Question (see Section 4),
- Boij-Söderberg Theory (see Sections 5 and 6).

Of course, there are a number of other interesting aspects of syzygies. The expos-



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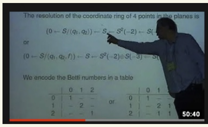

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Concrete Nonsense

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Boij–Söderberg theory I: preliminaries