Virtual Resolutions and Syzygies

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Polynomials come in different flavors…

Increasing structure →

Polynomials	**Graded Polynomials** or $\mathbb{Z}$-graded polynomials	**Multigraded Polynomials** or $\mathbb{Z}^r$-graded polynomials
$5xy + y^2 - x^3$	$\underbrace{x_0^2 + 3x_1x_3 - 6x_2^2}_{\text{degree} = 2 \in \mathbb{Z}}$	$\underbrace{x_0^3y_1^2 - 2x_0x_1^2y_0y_1}_{\text{degree} = (3,2) \in \mathbb{Z}^2}$
affine geometry	*projective geometry*	*toric geometry*

Goal: Move tools studying graded polynomials to the multigraded setting

Syzygies

Hilbert Syzygy Theorem (1890): If M is finitely generated, M has a *unique* minimal free resolution with length $\leq$ dim S

Example. $S = k[x_0, x_1, x_2]$ and $I = \langle x_0x_1, x_0x_2 \rangle$.

<u>Goal</u>: Study syzygies of $M = S/I$.

minimal free resolution

$S(-3)_i = S_{i-3}$

$$0 \longleftarrow M \xleftarrow{\;1\;} S \xleftarrow{(x_0x_1 \quad x_0x_2)} S(-2)^2 \xleftarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} S(-3) \longleftarrow 0$$

$\langle x_0x_1, x_0x_2 \rangle$ is the *module of 1st syzygies*

they give relations on the generator 1

$(1)x_0x_1 = 0$

$(1)x_0x_2 = 0$

$\langle (x_2, -x_1) \rangle$ is the *module of 2nd syzygies*

they give relations on the generators x_0x_1 and x_0x_2

$x_2(x_0x_1) - x_1(x_0x_2) = 0$

Syzygies

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Geometrically: If I defines some variety $V(I) \subseteq \mathbb{P}^n$, the minimal free resolution is bounded by dim $\mathbb{P}^n = n$

Syzygies

Main Point: All degrees and ranks of the syzygies are numerical invariants of M!

$\mathbb{Z}$**-graded polynomials**		**Projective space**
ranks of syzygies	⟷	flatness, degree, genus (for curves), dimension
degrees of syzygies	regularity ⟷	sheaf cohomology bounds
lots of linear syzygies	Green's N_p conditions ⟷	high degree embeddings

Green (1984), Green-Lazarsfeld (1985), Schreyer (1986), Voisin (2002, 2005), Aprodu (2003), Farkas-Kemeny (2014), …

Cox '95

$\mathbb{Z}^r$-graded polynomials		Toric geometry
$\mathbb{C}[\underbrace{x_0, x_1}, \underbrace{y_0, y_1}]$ degrees: (1,0) (0,1)		$\mathbb{P}^1 \times \mathbb{P}^1$
Collection of $\mathbb{Z}^2$-graded polynomials	⟷	toric subvariety
maximal ideals* $\langle bx_0 - ax_1, dy_0 - cy_1 \rangle$	⟷	points $[a : b] \times [c : d]$
Irrelevant ideal $\langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$	⟷	Ø, the empty set
syzygies $0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots$	⟷ ✗	e.g. geometric hilbert syzygy theorem fails

What goes wrong?

Example. Let $I = \langle x_0x_1, x_1y_0, x_0y_1, y_0y_1 \rangle$ be the ideal corresponding to the two points $[0:1] \times [0:1]$ and $[1:0] \times [1:0]$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Here's a minimal free resolution of S/I:

$$0 \longleftarrow S/I \longleftarrow S \longleftarrow \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1)^2 \\ \oplus \\ S(0,-2) \end{matrix} \longleftarrow \begin{matrix} S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \end{matrix} \longleftarrow S(-2,-2) \longleftarrow 0$$

The length of this is longer than $\dim(\mathbb{P}^1 \times \mathbb{P}^1) = 2$.

Essential issue: If you generate a minimal free resolution, you get algebraic structure that is *geometrically irrelevant*.

Virtual Resolutions

Example. Let $I = \langle x_0x_1, x_1y_0, x_0y_1, y_0y_1 \rangle$ be the ideal corresponding to the two points $[0:1] \times [0:1]$ and $[1:0] \times [1:0]$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

minimal free resolution
ker d^i/im $\mathrm{d}^{d+1} = 0$

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$$0 \longleftarrow S/I \longleftarrow S \xleftarrow{(x_0x_1 \quad x_1y_0 \quad x_0y_1)} \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1)^2 \end{matrix} \xleftarrow{\begin{pmatrix} -x_0 & -x_1 \\ y_0 & 0 \\ 0 & y_1 \end{pmatrix}} S(-2,-1)^2 \longleftarrow 0$$

homology
$H = I/\langle x_0x_1, x_1y_0, x_0y_1 \rangle$

Virtual Resolutions

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virtual resolution
"geometrically exact"
homology is supported on irrelevant ideal

$$0 \longleftarrow S/I \longleftarrow S \xleftarrow{\begin{pmatrix} x_0x_1 & x_1y_0 & x_0y_1 \end{pmatrix}} \begin{matrix} S(-2,0) \\ \oplus \\ S(-1,-1)^2 \end{matrix} \xleftarrow{\begin{pmatrix} -x_0 & -x_1 \\ y_0 & 0 \\ 0 & y_1 \end{pmatrix}} S(-2,-1)^2 \longleftarrow 0$$

homology
$H = I/\langle x_0x_1, x_1y_0, x_0y_1\rangle$

you can check:
$B^2H = 0$

Berkesch-Erman-Smith (2019)

Questions about Virtual Resolutions

1. *How can we construct examples of virtual resolutions?*

Theorem [Booms, C.] Here are some criteria for checking if an important family of complexes (generalized eagon-northcott complexes) are virtual.

2. *How can we measure how algebraically complicated virtual syzygies can get?*

Theorem [C.] A bound for curves.

① *How can we construct examples of virtual resolutions?*

Generalizing Koszul Complexes

Koszul Complex: $\varphi : S^n \xrightarrow{(s_1 \ \cdots \ s_n)} S$ gives a canonical complex

This complex tells you about $S/(s_1, \ldots, s_n)$

Eagon-Northcott Complex: $\varphi : S^n \xrightarrow{\begin{pmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & \ddots & \vdots \\ s_{m,1} & \cdots & s_{m,n} \end{pmatrix}} S^m$ gives a canonical complex

[Eagon, Northcott '62]

This complex tells you about $S/I_m(\varphi)$

$I_m(\varphi) = m \times m$ minors of φ

$$0 \to \begin{matrix} \Lambda^e E \\ \otimes \\ (S^{e-f-1}F)^* \otimes \Lambda^f F^* \end{matrix} \to \cdots \to \begin{matrix} \Lambda^{f+2} E \\ \otimes \\ F^* \otimes \Lambda^f F^* \end{matrix} \to \begin{matrix} \Lambda^{f+1} E \\ \otimes \\ \Lambda^f F^* \end{matrix} \to E \to F \to 0 \quad (\mathrm{EN}_1)$$

Eagon-Northcott Complex $\in$ Generalized Eagon-Northcott Complexes

$$0 \to \begin{matrix} \Lambda^e E \\ \otimes \\ (S^{e-f-2}F)^* \otimes \Lambda^f F^* \end{matrix} \to \cdots \to \begin{matrix} \Lambda^{f+2} E \\ \otimes \\ \Lambda^f F^* \end{matrix} \to \Lambda^2 E \to E \otimes F \to S^2 F \to 0 \quad (\mathrm{EN}_2)$$

① *How can we construct examples of virtual resolutions?*

Theorem [Booms, C.] Let $\varphi : F \to G$ be Pic(X)-graded map. Let $f = \text{rank } F$ and $g = \text{rank } G$ and B be the irrelevant ideal.
The generalized Eagon-Northcott complexes C^i are all virtual when depth($\langle$max'l minors of $\varphi\rangle : B^\infty) \geq f - g + 1$

Upshot: Virtual resolutions seemingly not coming from a larger minimal free resolution

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Upshot: First virtual resolutions not coming from a larger minimal free resolution

2. *How can we measure how algebraically complicated virtual syzygies can get?*

(2) *How can we measure how algebraically complicated virtual syzygies can get?*

Castelnuovo-Mumford Regularity

In algebraic geometry, you learn:

— the hilbert function of a projective variety X is eventually polynomial

$h_X(r) = \dim H^0(\mathscr{O}_X(r))$ agrees with some polynomial when $r >> 0$

— Serre vanishing: As long as $r >> 0$, a coherent sheaf $\mathscr{F}(r)$ on projective space is globally generated and its higher cohomology vanishes

⋮

Moral: Life is easier when we look past some magic number.

This magic number is the Castelnuovo-Mumford Regularity

② *How can we measure how algebraically complicated virtual syzygies can get?*

Castelnuovo-Mumford Regularity

Definition ['66 Mumford] The *Castelnuovo-Mumford regularity* of a coherent sheaf $\mathscr{F}$ on $\mathbb{P}^n$ are all the $r \in \mathbb{Z}$ satisfying

$$H^i(\mathbb{P}^r, \mathscr{F}(r-i)) = 0 \text{ for all } i > 0$$

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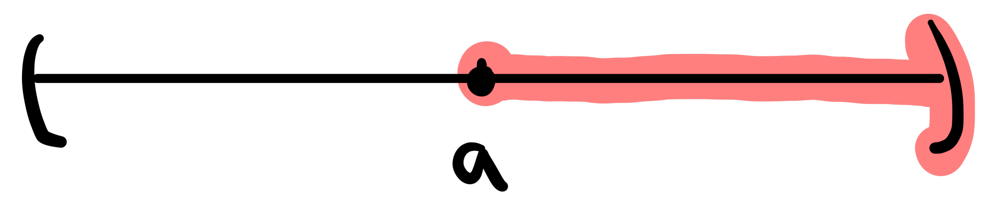
— this region is a cone in $\mathbb{Z}$ of the form $a + \mathbb{N}$.



— if n is in the regularity of I_X defining some variety X, then I_X is generated by polynomials of degree $\leq n$.

② *How can we measure how algebraically complicated virtual syzygies can get?*

Castelnuovo-Mumford Regularity

Example. Consider the twisted cubic

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[s : t] \mapsto [s^3, s^2t, st^2, t^3]$$

Minimal free resolution: $0 \to S(-3)^2 \to S(-2)^3 \to I_X \to 0$ Eagon-Northcott of $\varphi = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$

Castelnuovo-Mumford regularity is the “width”, or how quickly the degrees of the syzygies are rising.

② *How can we measure how algebraically complicated virtual syzygies can get?*

Castelnuovo-Mumford Regularity

"Most important theorem on regularity to date"

- Eisenbud (The Geometry of Syzygies, 2002)

Theorem [Gruson, Lazarsfeld, Peskine '83]: Let $C \subseteq \mathbb{P}^r$ be a smooth curve of deg d. Then $d + 2 - r + \mathbb{N} \subseteq$ reg C.

Example.

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$[s : t] \mapsto [s^3, s^2t, st^2, t^3]$$

degree is 3, $r = 3 \Longrightarrow 2 + \mathbb{N} \subseteq$ reg $C \Longrightarrow I_X$ generated by degree 2 polynomials

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Castelnuovo-Mumford Regularity

Theorem [Gruson, Lazarsfeld, Peskine '83]: Let $C \subseteq \mathbb{P}^r$ be a smooth curve of deg d. Then $d + 2 - r + \mathbb{N} \subseteq$ reg C.

Why is this important?

— settled and generalized classical work of Mumford, Castelnuovo, …

— gave progress towards a larger program of study (Eisenbud-Goto conjecture)

— The ideas that went into the proof

② *How can we measure how algebraically complicated virtual syzygies can get?*

Ideas in Proof of

Theorem [Gruson, Lazarsfeld, Peskine '83]: Let $C \subseteq \mathbb{P}^r$ be a smooth curve of deg d. Then $d+2-r+\mathbb{N} \subseteq \operatorname{reg} C$.

– A map $X \to \mathbb{P}^r$ is given by a globally generated line bundle $\mathscr{L}$ on X
globally generated $\Longrightarrow$ surjection $V \otimes \mathcal{O}_X \to \mathscr{L} \to 0$ where $V \subseteq H^0(X, \mathscr{L})$

$$0 \to \mathscr{M}_V \to V \otimes \mathcal{O}_X \to \mathscr{L} \to 0$$

Insight 1: Syzygies of $X \to \mathbb{P}^r$ are controlled by cohomology of $\mathscr{M}_V$

See [Green '84] for a bigger picture

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Insight 1: Syzygies of $X \to \mathbb{P}^r$ are controlled by cohomology of $\mathscr{M}_V$

– Imagine you have a presentation

$$\mathscr{E} \xrightarrow{\varphi} \mathscr{F} \longrightarrow I_X \longrightarrow 0$$

Insight 2: Eagon-Northcott complexes of φ can tell you things about I_X

② *How can we measure how algebraically complicated virtual syzygies can get?*

Ideas in Proof

Theorem [Gruson, Lazarsfeld, Peskine '83]: Let $C \subseteq \mathbb{P}^r$ be a smooth curve. Suppose $\mathscr{A}$ is a line bundle on C such that

$$H^1(\mathbb{P}^r, \wedge^2 \mathscr{M}_{\mathscr{L}} \otimes \mathscr{A}) = 0 .$$

Then we get a resolution

$$\cdots \longrightarrow \bigoplus \mathscr{O}_{\mathbb{P}^r}(-a-1) \to \bigoplus \mathscr{O}_{\mathbb{P}^r}(-a) \to I_C \to 0.$$

where $a = \dim H^0(\mathbb{P}^r, \mathscr{A})$.

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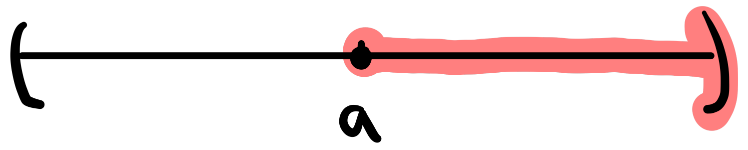
$\Downarrow$ Facts relating resolutions to regularity

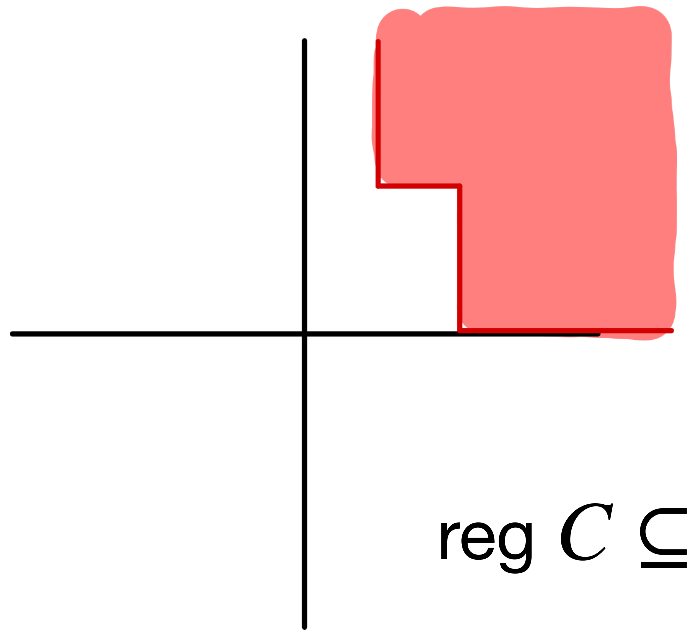
Corollary [Gruson, Lazarsfeld, Peskine '83]: Let $C \subseteq \mathbb{P}^r$ be a smooth curve of deg d. Then $\operatorname{reg}(\mathscr{O}_{\mathbb{P}^r}(-d-2+r)) \subseteq \operatorname{reg}(I_C)$.

② *How can we measure how algebraically complicated virtual syzygies can get?*

Generalizing to $\mathbb{P}^{\vec{r}} = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$



reg$(C) \subseteq$ *Pic*$(\mathbb{P}^r) = \mathbb{Z}$



reg $C \subseteq$ Pic$(\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}) = \mathbb{Z}^n$

Castelnuovo-Mumford regularity

Multigraded regularity
['07 Maclagan, Smith]

$\mathcal{O}_{\mathbb{P}^r}(-a)$ ⇢ $\mathcal{O}_{\mathbb{P}^{\vec{r}}}(-\vec{a})$

② *How can we measure how algebraically complicated virtual syzygies can get?*

Generalizing to $\mathbb{P}^{\vec{r}} = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$

Theorem [C.]: Let $p : C \hookrightarrow \mathbb{P}^{\vec{r}}$ be a smooth curve. Suppose $\mathscr{A}$ is a line bundle on C such that

$$H^1(\mathbb{P}^{\vec{r}}, p^*\Omega^{\vec{m}}_{\mathbb{P}^{\vec{r}}}(\vec{m}) \otimes \mathscr{A}) = 0 \text{ for all } m_1 + \cdots + m_n = 2$$

Then we get a resolution

$$\cdots \longrightarrow \mathscr{E}_1 \to \mathscr{E}_0 \to I_C \to 0.$$

Furthermore, this resolution is "linear" in some sense.

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$$\Downarrow$$ New facts relating resolutions to regularity

Corollary: reg$(\mathscr{E}_0) \subseteq$ reg(I_C).

② *How can we measure how algebraically complicated virtual syzygies can get?*

Generalizing to $\mathbb{P}^{\vec{r}} = \mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_n}$

Example. Consider this map:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$$

$$[s:t] \mapsto [t^2s - 4s^3 : t^3 - 4s^2t : t^2s - 3s^2] \times [s^2t - t^3 : s^3 - st^2 : t^3]$$

This has degree (3,3). The theorem says:

$$(4{,}4) + \mathbb{N}^2 \subseteq \operatorname{reg}(I_C)$$

So I_C is generated by equations of degree $\leq (4{,}4)$

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Crux of proof: Show that the following is exact:

$$\bigoplus_{|\vec{m}|=1} \mathscr{O}_{\mathbb{P}^{\vec{r}}}(-\vec{m})^{\oplus h^0_{\mathscr{A}}(\vec{m})} \xrightarrow{\varphi} H^0(X, \mathscr{A}) \otimes \mathscr{O}_{\mathbb{P}^{\vec{r}}} \longrightarrow p_*\mathscr{A} \longrightarrow 0$$

$h^i_{\mathscr{A}}(\vec{m}) :=$ dim $H^i(X, p^*\Omega^{\vec{m}}_{\mathbb{P}^{\vec{r}}}(\vec{m}) \otimes \mathscr{A})$

The resolution in the theorem is the Eagon-Northcott of φ

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$$C \times \mathbb{P}^{\vec{r}} \xrightarrow{p \times \text{id}} \mathbb{P}^{\vec{r}} \times \mathbb{P}^{\vec{r}}$$

The resolution in the theorem is the Eagon-Northcott of φ

Step 1: Create a resolution K of the graph of p inside $\Gamma_p \subset C \times \mathbb{P}^{\vec{r}}$ $\quad K = (p \times \text{id})^*(B) \longleftarrow B = \boxtimes_{i=0}^{n} B_i$

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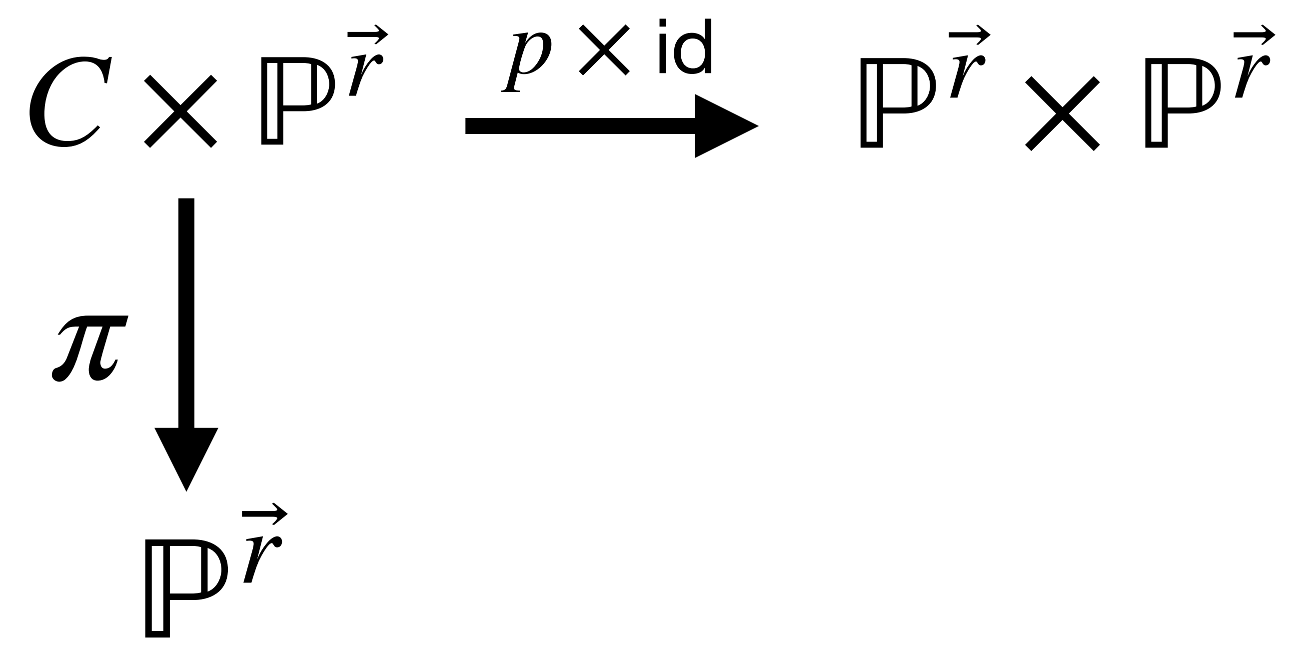

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Step 2: Compute the derived pushforward $\mathbf{R}\pi_*(K \otimes \pi^*\mathscr{A})$

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The resolution in the theorem is the Eagon-Northcott of φ

$$\begin{array}{ccc} C \times \mathbb{P}^{\vec{r}} & \xrightarrow{p \times \mathrm{id}} & \mathbb{P}^{\vec{r}} \times \mathbb{P}^{\vec{r}} \\ \downarrow \pi & & \\ \mathbb{P}^{\vec{r}} & & \end{array}$$

Step 1: Create a resolution K of the graph of p inside $\Gamma_p \subset C \times \mathbb{P}^{\vec{r}}$

Step 2: Compute the derived pushforward $\mathbf{R}\pi_*(K \otimes \pi^*\mathscr{A})$

Corollary. Let $C \subseteq \mathbb{P}^{\vec{r}}$ be a integral nondegenerate curve of degree **d** and define $a := \max\{d_i + d_j - r_i - r_j \mid i \neq j\} + 2$. Then $(\min\{d_k + ar_k, a\})_k + \mathbb{N}^n \subseteq \operatorname{reg} I_C$.

Thanks!